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What is functional thinking? Using cosine similarity matrix in a semantic ontological analysis

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ABSTRACT

Knowing functions and functional thinking have recently moved from just knowledge for older students to incorporating younger students, and functional thinking has been identified as one of the core competencies for algebra. Although it is significant for mathematical understanding, there is no unified view of functional thinking and how different aspects of the concept are used as a theoretical base. In this paper, we analyse different definitions used in empirical studies. First, we did a systematic research review resulting in 19 empirical studies focusing on functional thinking with an appropriate theoretical underpinning. The definitions were analysed using an AI tool. After that, we analysed the results using intrinsic mathematical properties of how functions can be defined in mathematics to identify core aspects of the definitions. According to the analysis, two definitions capture most of the key aspects of functional thinking, and most empirical studies use these key concepts. These two definitions treat functional thinking as products or products and processes. One definition used in one empirical study stands out by theoretically operationalizing functional thinking as a process. As such, different ontological assumptions are made in the studies; however, in some cases, having the same epistemological outcome. From a methodological point of view, the cosine similarity matrix was a useful tool for an ontological analysis, but a qualitative analysis is still needed to make meaning of it.

Keywords: cosine similarity matrix, functional thinking, functions, large language models, ontological analysis

INTRODUCTION

Functions and their relationship to algebra have gained increasingly more attention: we can see it from research focusing on algebraic thinking (e.g., Chimoni et al., 2023), including functional thinking as a means for learning algebra (e.g., Blanton & Kaput, 2011), as well as studies focusing only on functional thinking (e.g., Moss et al., 2020). Despite increasing attention, Ding et al. (2023) conclude that functional thinking in mathematics has not been addressed systematically. As a mathematical concept, 'function' is one of the central ones (Häggström, 2005): "The concept of a function is fundamental to virtually every aspect of mathematics and every branch of quantitative science" (Warren et al., 2013, p. 76). Functional thinking can be seen as part of algebraic thinking (e.g., Kaput, 2008), and studies have shown it predicts and explains algebraic thinking to a larger degree than modelling and generalized arithmetic (Chimoni et al., 2023). Previously, research has mainly focused on older students' understanding and reasoning about functions (e.g., Veith et al., 2023), but studies show that younger students are capable of operating and working with functions, including that working with students' functional thinking allows them to approach algebra more easily (Blanton & Kaput, 2011). One example of such a study is Blanton et al. (2017), who studied grade 1 students' functional thinking. They showed that the students could understand variables and different notations and how to represent functional relationships. Hence, functional thinking could be part of mathematics education from an early age. Although there has been much attention to algebraic thinking (e.g., Veith et al., 2023), surprisingly, few studies have focused on how functional thinking is defined, interpreted, and treated in mathematics education research. Such understanding lies under the area of ontology, which can be defined as an explicit description of conceptualisation (Obitko et al., 2004). An analysis of that kind provides information about how a concept is a shared understanding but also helps to see patterns in the theoretical structure of a concept. This is of interest since how a concept is defined and interpreted will influence how one can theoretically operationalise it in different studies (Mason, 2018). As such, the results are dependent on epistemology. In addition, a recent review shows that most mathematics education research focusing on algebra, including the ones on functional thinking, comes from the same country (the USA) and the same authors (Veith et al., 2023). Given that there are many different teaching traditions on how functions and functional thinking are treated (e.g., Ambrus et al., 2018), it is plausible to assume that there could be a variation when looking at how functional thinking is theoretically operationalized. Therefore, a more precise analysis could shed information on how research in functional thinking part of the same non-pluralistic body of knowledge is as in algebra (e.g., Veith et al., 2023) or if other structures exist.

The present study aims to investigate the concept of ‘functional thinking’ for younger students by doing an ontological analysis of data from a systematic literature review. The research questions are:

RQ1. How are the different definitions used in empirical research related to each other?

RQ2. What are the key concepts in different definitions of functional thinking?

The present paper’s subordinate aim is to develop and test machine learning and large language models as tools for ontological analysis.

BACKGROUND

Starting with functions, mathematically, one can treat functions in three ways (Kleiner, 1989). The two most famous mental images are the geometric one (i.e., to see a function as expressed by a curve) or to treat it as an algebraic expression. The third one is to treat it as a correspondence, where the mental image is to see it as an ‘input-output machine’. Building on the latter, one common mathematical definition of functions is that a function from a set *A* to a set *B* assigns to each element of *A* exactly one element in *B*, where *A* is the domain of the function and set *B* is the codomain of the function (Häggström, 2005). In this sense, a function is a mapping (Eq. 1):

$$f: A \rightarrow B. \tag{1}$$

Historically, the concept function has been a challenge for mathematicians (Häggström, 2005). Some of the most famous mathematicians have been involved in developing the concept (Kleiner, 1989). In an overview of the concept, González-Polo and Castaneda (2024) concluded that this change was substantial, where functions moved from a motion-based interpretation to a more algebraic modern definition. This change included both how function was expressed, and which representations were used, as well as how it was interpreted and applied (for a more comprehensive description of how the concept function has changed, see Kjeldsen & Lützen, 2015). From a mathematical education perspective, it is not just the concept in itself that captures the interest but also how students (and others) develop functional thinking and what such thinking entails (e.g., Blanton et al., 2017). Given the different mental images of functions (e.g., González-Polo & Castaneda, 2024; Kleiner, 1989), it is not surprising that students’ functional thinking is not a straightforward business. Some struggles are connected to how functions are treated in school mathematics (Beeman et al., 2024). Already in 1908, in the first version of Klein’s (2016) seminal book on mathematics in school, he concludes that there is a discontinuity in how the concept function is treated in school mathematics.

> The teacher manages to get along still with the cumbersome algebraic analysis, in spite of its difficulties and imperfections, and avoids *any smooth infinitesimal calculus*, although the 18th century shyness toward it has long lost all point. The reason for this probably lies in the fact that *mathematical teaching in schools and the advance of research lost all touch with each other after the beginning of the 19th century*. And this is the stranger since the specific training of future teachers of mathematics dates from the early decades of that century. I called attention in the preface to this *discontinuity*, which was of long standing, and which impeded every reform of the school tradition: In the schools, namely, one cared little whether and how the approaches taught might be extended within higher education and one was therefore satisfied often with definitions which were perhaps sufficient for the present, but which failed to meet more far-reaching demands (Klein, 2016, p. 168; italics as in original).

As such, Klein (2016) does not support the idea that algebra early should be something else compared to early algebra (e.g., Carraher & Schliemann, 2007). Given how the concept is treated, which to some degree reflects the historical development of the concept function (e.g., González-Polo & Castaneda, 2024; Kjeldsen & Lützen, 2015), the education of functions may create gaps or discontinuity as Klein (2016) puts it. Empirical studies on students’ understanding of aspects of functions, such as Juter (2012) and Afriyani et al. (2018), support the above observation. Using Klein’s (2016) concept of discontinuity, we would like to raise the issue of whether this could be the case for research in mathematics education on functional thinking too. Some struggles might stem from how researchers ontologically and epistemologically frame the concepts and the related concept of functional thinking. Given how we treat mathematical concepts, we are part of the history that shapes the concepts, and the concept ‘function’ has, through history, had many different shapes (Häggström, 2005).

Moving on to functional thinking, although Ding et al. (2023) conclude that functional thinking in mathematics has not been addressed systematically, it has been a central concept for mathematics education for many years. According to Gutzmer (1908) (as cited in Vollrath, 1986), functional thinking is an ability that has been a key concept since the Meran Conference in 1905. This ability is defined as having two characteristics (Vollrath, 1986). The first one is that “dependences between variables can be stated, postulated, produced, and reproduced” (Vollrath, 1986, p. 387). It focuses on the mathematical content (i.e., dependencies between variables), and it should be handled in different ways. It is similar to the definition presented by Kaput (2008), a commonly used one, where functional thinking is seen as part of algebraic thinking and is interpreted as the generalization of relationships between covarying quantities. The second characteristic is that “assumptions about the dependence can be made, can be tested, and if necessary can be revised” (Vollrath, 1986, p. 387). It can be interpreted as mathematical reasoning with arguments justified using intrinsic mathematical properties (Sumpter, 2016) and, therefore, can be considered a part of mathematical thinking (e.g., Burton, 1984). In combination, these two characteristics cover a variety of concepts and processes that are connected to the understanding of the mapping as described in Eq. (1), for instance, variables, co-variation, and correspondence, as well as generalizations, justifications, and different types of reasonings (Chimoni et al., 2018). Or as Georges (1946) concluded that “functional thinking is concerned in recognition, rationalization, and manipulation of relationships between quantities” (p. 736). Then, functional thinking is the ability to understand and manipulate the relationship between variables (Vollrath, 1986).

One issue that can also be a strength is the development of a theoretical framework. For example, we take a development observed by Kieran (2022) in her review paper on early algebraic thinking. She noticed that Stephens et al. (2017) used a revised version of a curricular unit by Blanton et al. (2015) that aims to promote functional thinking related to pattern and correspondence. Then, just as in a relay, Pang and Sunwoo (2022) pick up the work of Stephens et al. (2017) and refine the work further. The strength of such development is that the relation theory-empirical data can become more precise and valid. The problems, however, can be if adjustments lose some intrinsic mathematical properties, such as properties that might not be visible in, for instance, early algebra but present in university algebra, hence increasing discontinuity as established by Klein (2016). One researcher who has made such an important note is Häggström (2005) but concerning functions. He concluded that some definitions of functions state that A and B are domains of numbers, which is a limitation that restricts the possibility of making further abstractions. Using such framing means that researchers might miss important results, with implications that teaching using those research results has the effect that students are limited in developing their functional thinking.

As stated earlier, most empirical research in functional thinking has previously been on older students (Veith et al., 2023). One explanation for the delay of research on younger students is that algebra can only follow arithmetic, meaning that mathematics has been treated as a compilation of isolated topics that are dealt with in a specific order (Chimoni et al., 2018). Today, algebra – including functions–is a core part of many countries’ school curriculums already at an early age (Chimoni et al., 2023). It reflects the central role of functions in mathematics, both as a school subject and as a mathematical topic, now and throughout history (Häggström, 2005). The research includes studies on how young students can reason about functions, including how the progression of functional thinking could look (e.g., Blanton et al., 2017; Inhelder & Piaget, 1958; Vollrath, 1986). Now, more studies illustrate how functional thinking is part of algebraic thinking (e.g., Kaput, 2008), and how different types of reasoning can predict various aspects of algebraic thinking. One recent study by Chimoni et al. (2023) used structural equation modelling to show that spatial reasoning predicts students’ abilities in functional thinking, including modelling. In addition, deductive reasoning was also linked to functional thinking, but it was predicting the other two strands of algebraic thinking (e.g., Kaput, 2008) as well. The results also showed that when functional thinking was compared to generalized arithmetic, all grades (grade 4-grade 7) had the same pattern (Chimoni et al., 2023). The conclusion is that functional thinking demands deductive and spatial reasoning independent of grades, meaning that the two characteristics listed by Vollrath (1986) are valid both for younger and older students.

METHODS

The first step was to do a systematic search of research in mathematics education. We did a search in ERIC (ProQuest) using the words ‘functional thinking’ and ‘mathematics’ and limited the search to the years 2002-2022. It generated 39 papers. As a comparison, a similar search on Scopus generated 47 papers. Since we were restricted in the number of papers we could work on within the analysis, we opted for the results from the ERIC search. A first screening, reading titles and abstracts to remove papers that did not focus on functional thinking and younger students, resulted in 25 papers. One of the papers that was removed at this stage was a paper that focused only on gifted students, which was outside the scope of the study. The next step was to read entire papers with three conditions: they had to be explicitly peer-reviewed, they needed to have a definition of functional thinking, and the paper required to treat it as the concept in the focus of the analysis. The reading resulted in 19 papers; see **Table 1**.

Table 1. The context for the data to be analyzed

Reference	Title	Definition
Afonso and McAuliffe (2019)	Children’s capacity for algebraic thinking in the early grades	This provides a rich context for developing algebraic thinking practices involving generalizing and reasoning, representing and justifying functional relationships, a way of helping children to see and describe mathematical structures and relationships and so construct meaning.
Blanton et al. (2015)	The development of children’s algebraic thinking: The impact of a comprehensive early algebra intervention in third grade	Involves generalizing relationships between covarying quantities and representing and reasoning with those relationships through natural language, algebraic notation, tables, and graphs.
Chimoni et al. (2018)	Examining early algebraic thinking: Insights from empirical data	The generalization of relationship between co-varying quantities: This strand was related to the ability for expressing numerical and figural patterns as functions and algebraic expressions.
Moss et al. (2020)	Representations and conceptions of variables in students’ early understandings of functions	Representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances.
Pang and Sunwoo (2022)	Design of a pattern and correspondence unit to foster functional thinking in an elementary mathematics textbook	Generalizing relationships between covarying quantities, expressing those relationships in words, symbols, tables, or graphs, and reasoning with these various representations to analyze function behavior.
Pinto et al. (2022)	Functional relationships evidenced and representations used by third graders within a functional approach to early algebra	Focus on the relationship between two (or more) variables; specifically, the types of thoughts that go from specific relationships to generalizations of relationships.

Table 1 (Continued). The context for the data to be analyzed

Reference	Title	Definition
Pittalis et al. (2020)	Young students' functional thinking modes: The relation between recursive patterning, covariational thinking, and correspondence relations	The type of thinking that focuses on the invariant relation between two varying quantities/variables, Involves noticing, generalizing, and abstracting relations between covarying quantities/variables; representing these relations through natural language, symbols, and appropriate representations; and using the generalized representations in problem-solving situations.
Pitta-Pantazi et al. (2020)	Different types of algebraic thinking: An empirical study focusing on middle school students	One looks for ways to express a systematic variation of instances and involves the idea of causality, growth, and continuous joint variation.
Ramírez et al. (2022)	Structures and representations used by 6th graders when working with quadratic functions	Focuses on the relationship between two (or more) co-varying quantities. It concerns the process that leads from the relationship of specific cases to generalizations of that relationship.
Somasundram et al. (2019)	Pattern generalization by year five pupils	Incorporate building and generalizing patterns and relationships using diverse linguistic and representational tools and treating generalized relationships, or functions, that results as mathematical objects useful in their own right.
Sproesser et al. (2022)	Changing between representations of elementary functions: Students' competencies and differences with a specific perspective on school track and gender	Is characterized as a specific and meaningful way of thinking in relationships, interdependencies, and changes.
Stephens et al. (2017)	A learning progression for elementary students' functional thinking	The process of building, describing, and reasoning with and about functions.
Tanışlı (2011)	Functional thinking ways in relation to linear function tables of elementary school students	Representational thinking that focuses on the relationship between two (or more) varying quantities; a process of building, describing, and reasoning with and about functions; ability to see the relationships between two data sets.
Warren and Cooper (2005)	Introducing functional thinking in year 2: A case study of early algebra teaching	It is a matter of relation and transformation together with how the value of certain quantities relates to the value of other quantities.
Warren et al. (2006)	Investigating functional thinking in the elementary classroom: Foundations of early algebraic reasoning	How certain quantities relate, or change or transformed, to other quantities; Representational thinking that focuses on the relationship between two or more varying quantities.
Warren et al. (2013)	Exploring young students' functional thinking	Perceptual act of noticing generalities.
Wilkie and Ayalon (2018)	Investigating years 7 to 12 students' knowledge of linear relationships through different contexts and representations	Representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances.
Wilkie and Clarke (2016)	Developing students' functional thinking in algebra through different visualizations of a growing pattern's structure	Representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances.
Xolocotzin et al. (2022)	Starting points: Understanding children's pre-instructional intuitions about function tables	Building and generalizing patterns and relationships, using diverse linguistic and representational tools and treating generalized relationships, or functions, that result as mathematical objects useful in their own right.

In **Table 1**, besides presenting each reference, we also present the different definitions of functional thinking used, constituting the data for the present paper. The amount was considered enough to do the analysis and given that the sub-aim of the paper is to test new methods of analysis, the data set could not be too big to handle for a qualitative analysis. At the same time, it is crucial that the dataset is reasonably comprehensive for the modelling (Zhu et al., 2023). A first reading showed that there were only a few authors that were repeated, and a closer reading revealed that the authors have sometimes changed theoretical framing, which meant that the conclusion was that the dataset was appropriate for an ontological analysis. The limitations, however, are that although comparing the search from ERIC to Scopus obtaining similar results, some papers were inevitably not included. In addition, given the aim to study research on younger students meant that several papers were removed since they studied older students, meaning that an analysis of these papers might generate different results. Despite these limitations, the conclusion was that the data was enough with respect to the aim of the study.

The next step was to make an ontological analysis of the definitions provided in the papers. One way of making an ontological analysis is to do a formal concept analysis (FCA). It is a theory of data analysis that aims to identify conceptual structures among data sets (Obitko et al., 2004). The analysis works in three steps. The first step is to see concepts as described by properties (e.g., Lithner, 2008). For us, it means that keywords are identified and treated as intrinsic mathematical properties. Using Kaput's (2008) definition as an example, the properties are 'generalization', 'covarying', and 'quantities', but also relationships between these words, for instance, to treat 'covarying quantities' as one word. The second step is to determine the hierarchy of concepts (Obitko et al., 2004). It means that one studies the properties of each concept, and the properties decide the hierarchy. It means that if 'covarying quantities' is used more than the words separately, then the unit 'covarying quantities' has a higher order than 'covarying'. It also includes which properties are more important in a definition (e.g., Häggström, 2005), for instance, to determine if 'covarying quantities' is more important than 'generalization'. This is decided by studying the relationship between different definitions, that is, how close the properties are.

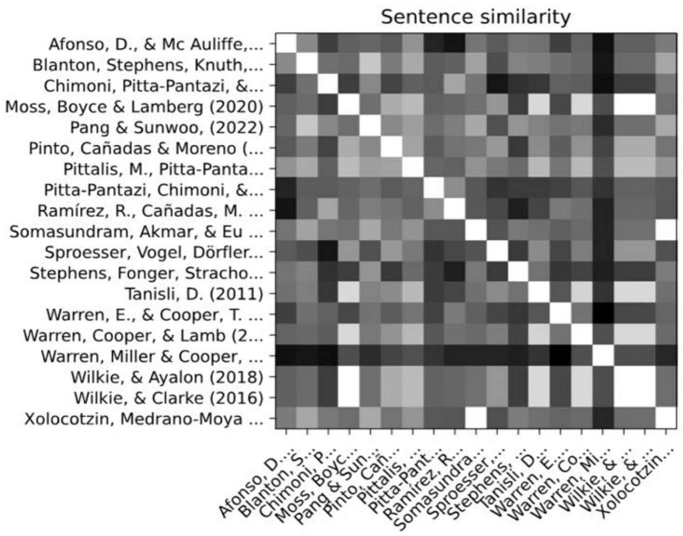

Figure 1. Relationships between different definitions. Whitness indicate strenghtness of cosine similarity. (Source: Authors' own elaboration)

The third step is to seek similarities to identify duplicates and see to what degree different definitions are dissimilar. The rule of identity is that if properties of different concepts are the same, then concepts are the same–they have the same identity, including all relationships (Obitko et al., 2004). Given the entrance of AI, it implies that the analysis can be done to a precise semantic level. To compare the nineteen definitions, we created sentence embeddings for each of them using the Hugging Face sentence-transformers library (https://huggingface.co/sentence-transformers/all-MiniLM-L6-v2). Embedding involves assigning a number to each sentence, where similar sentences receive close-by numbers. The sentence transformers map text to a 384-dimensional vector space, whereby similar vectors have similar usage in English. We then calculated the cosine similarity (a distance measure) between the vectors representing each definition, which gives a cosine similarity matrix. The method is appropriate when representing objects in machine learning methods (Sidorov et al., 2014), but to our knowledge, it is not common in mathematics education research.

In the cosine similarity matrix, whiter-shaded are more similar, while darker-shaded pairs are less similar. When finding pairs that are very similar (white) and not similar (black), we can analyze the relationship between different definitions, that is, how close the properties are (e.g., Obitko et al., 2004). Then, we take the definitions that are the most similar and the least similar to identify the key concepts in these definitions. The key concepts are then analysed using the two characteristics of functional thinking described by Vollrath (1986). The results will reveal patterns in which mathematical properties are emphasised as mathematical content (i.e., the first characteristic listed by Vollrath, 1986) and processes (i.e., the second characteristic listed by Vollrath, 1986). Such analysis provides information beyond the similarity of words (c.f., Sidorov et al., 2014) and gives meaning to FCA (e.g., Obitko et al., 2004). The method was first tested in a pilot study (Blomqvist & Sumpter, 2024).

RESULTS

The results are presented first with the cosine similarity matrix, and then we present the results of the qualitative interpretations. Starting with the first analysis it generated the following cosine similarity matrix, see **Figure 1**.

As shown in **Figure 1**, the matrix shows a reasonably pluralistic pattern. It means several definitions are used, each with various key concepts with some overlaps and differences. Taking definitions that have been connected using the same material, namely Blanton et al. (2015), Stephens et al. (2017), and Pang and Sunwoo (2022), the analysis shows that they are founded in two different theoretical framings. The first definition in this chain is:

> Involves generalizing relationships between covarying quantities and representing and reasoning with those relationships through natural language, algebraic notation, tables, and graphs (Blanton et al., 2015, p. 43).

The second definition is the one from Stephens et al. (2017):

> The process of building, describing, and reasoning with and about functions (p. 144).

The third definition and the last one in this chain is:

> Generalizing relationships between covarying quantities, expressing those relationships in words, symbols, tables, or graphs, and reasoning with these various representations to analyze function behavior (Pang & Sunwoo, 2022, p. 1315).

Comparing Blanton et al. (2015) with Stephens et al. (2017), the grey scale is darker compared to Blanton et al. (2015) and Pang and Sunwoo (2022). Similar results are obtained when comparing Stephens et al. (2017) and Pang and Sunwoo (2022). Hence, according to the cosine similarity matrix, there are more significant differences between the second definition compared to the first one and the third one. Looking closer at the mathematical properties used in the three definitions, Blanton et al. (2015) and Pang and Sunwoo (2022) list more processes and products. The definition suggested by Stephens et al. (2017) talks on a general level about the mapping presented in Eq. (1). Hence, there is no chronological pattern between these three definitions; that is, the grey scale becomes more or less intense. Further investigations show that Stephens et al. (2017) rely on the work of Carraher and Schliemann (2007), hence having a different theoretical framing than the other two studies, although using the same materials. Although different ontological framing, the epistemological outcome appears to be consistent.

Looking at the two extremes, two pairs of white stood out from the diagonal. The first pair are Moss et al. (2020) on one side and Wilkİe and Ayalon (2018) together with Wilkie and Clarke (2016) on the other side. They are using the definition from Smith (2008):

> Representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances (p. 143).

Functional thinking is then about how to represent the mapping presented in Eq. (1), and the ability to generalize the relationship between the domain *A* and the codomain *B*. This definition has many key concepts that are shared by the other references. The grey scale means that the whiter shade of grey, the more key concepts are in common. According to the results presented in the cosine similarity matrix (**Figure 1**), the definition used by Moss et al. (2020) shares several key concepts with many of the other studies in the analysis.

The other pair are Somasundram et al. (2019) and Xolocotzin et al. (2022). They are using the same definition of functional thinking, beside the word 'incorporate', as

> incorporate building and generalizing patterns and relationships using diverse linguistic and representational tools and treating generalized relationships, or functions, that result as mathematical objects useful in their own right (Blanton & Kaput, 2011, p. 8).

Functional thinking is both processes (building and generalizing patterns and relationships) and products (mathematical objects). The intrinsic mathematical properties are in the description of the processes: comparing Eq. (1) with this definition, we see that the mapping is represented in the generalized patterns and relationships, as well as the transformations connected to these entities.

The reference that has a definition least similar with the other ones, we find Warren et al. (2013). Here, we can identify a chain. Looking at this text and comparing it with Warren and Cooper (2005) and Warren et al. (2006), a shift has happened from the first two publications to the later one. Starting with Warren and Cooper (2005) and Warren et al. (2006), they define functional thinking as a matter of relation and transformation together with how the value of certain quantities relates to the value of other quantities. The reference is Chazan (1996), and further reading of Chazan (1996) shows that neither is the definition present in his text, nor does it aim to provide an ontological or epistemological understanding of algebraic thinking in general and functional thinking specifically. The conclusion is that the definition used by Warren and Cooper (2005) and Warren et al. (2006) should be seen as the authors' interpretation of Chazan (1996). The definitions of functional thinking used in these two texts (i.e., Warren & Cooper, 2005 and Warren et al., 2006) have several key concepts such as 'relation', 'transformation', and 'value of certain quantities'. Other authors have identified these concepts as core aspects of functional thinking (e.g., Kaput, 2008). It is visible in the cosine similarity matrix (**Figure 1**) that both texts, Warren and Cooper (2005) and Warren et al. (2006), have a relatively lighter greyscale, indicating relationships with other studies.

Looking at Warren et al. (2013), functional thinking is now defined as the perceptual act of noticing generalities. The authors move from one ontological standpoint of functional thinking to another. The definition comes from Radford (2006). When reading Radford (2006), functional thinking is described from a semiotic perspective and should be interpreted as a semiotic act, including bodily gestures. As we can see in **Figure 1**, the results of the analysis–that is, on a semantic level, comparing this definition with all the other definitions that have been used–show that it is the one that has the fewest key concepts in common with all the other definitions. In **Figure 1**, darker lines, horizontal and vertical, are visible in the cosine similarity matrix. Since it is not clear what the perceptual act involves, that is, what types of processes are involved in it, the researcher needs to make assumptions about what they might be with respect to functional thinking. Regarding concepts, the word 'generalizations' is true for all three strands of algebraic thinking; besides functional thinking, it is also used when describing generalized arithmetic and modelling (e.g., Kaput, 2008). Therefore, what is unique for functional thinking as a perceptual act of noticing generalities concerning the mapping described in Eq. (1) is unclear here.

DISCUSSION

As a summary, the results from the ontological analysis, here in the shape of FCA (Obitko et al., 2004,) allow us to identify semantic patterns. Three definitions stand out in the results. The first two are Smith's (2008) and Blanton and Kaput's (2011). The key concepts in these two definitions are a generalization of relationships and patterns, including the ability to represent the mapping between domain *A* and codomain *B*. Hence, the definitions are close to the mathematical definition of functions (e.g., Häggström, 2005). They also follow the historical development of the mathematical concept of function (e.g., Kleiner, 1989). The results showed that the definitions of the other empirical studies in this review mostly used the same key concepts such as 'relation', 'transformation', and 'value of certain quantities', all core aspects of functional thinking (e.g., Kaput, 2008). However, given that functional thinking demands deductive and spatial reasoning, independent of grades (Chimoni et al., 2023), one could raise the question of whether the definitions from Smith's (2008), and Blanton and Kaput's (2011) are more fruitful, given that they stress the generalization of relationships and patterns, including the mapping between domain *A* and codomain *B*. They could avoid the discontinuity that Klein (2016) warned us about +100 years ago. Using the results on how different definitions of functions can limit students' understanding of functions (Häggström, 2005), the same warning can be obtained for functional thinking. If the definition is restricted to, for instance, only patterns and generalizations about numbers and not allowed to include elements such as vectors (lengths), the transfer to understanding functions as motions will be a more significant step. The suggested implication is supported by studies such as the one by Ambrus et al. (2018), which shows how differently functions and functional thinking are and have been treated in school mathematics, impacting what students have the opportunity to learn.

The exception in the results is Warren et al. (2013), who wrote their definition with reference to Radford (2006). Not only does this definition differ, but it also includes an epistemological shift: from treating it as a product to construing it as a process. The shift has some implications. A closer analysis of the theoretical underpinnings of the methodology of Warren et al. (2013), it is not clear how functional thinking as a process differs from other algebraic thinking (or other mathematical thinking) from a mathematical point of view (i.e., intrinsic and relevant mathematical properties (e.g., Lithner, 2008). Although capturing a process might be more successful concerning the 'thinking' part (e.g., Vollrath, 1986), it is not as strict concerning the 'functional' part. However, to make such a sharp conclusion, one needs to analyze other studies using the work of Radford (2006). A closer analysis shows that the definition provided by Warren et al. (2013) is more of a synthesis of Radford (2006), but further work is needed. One paper is not enough. We therefore suggest this as an appropriate topic for further research.

Comparing the definitions used in the different papers with the description of functional thinking as Vollrath (1986) presents, there is more emphasis on the mathematical content with respect to dependences, and little on the second characteristic, that "Assumptions about the dependence can be made, can be tested, and if necessary can be revised." (Vollrath, 1986, p. 387). As such, the empirical studies do not cover the reasoning part of functional thinking (e.g., Sumpter, 2016) or, in the wider term, mathematical thinking (e.g., Burton, 1984). Our conclusion is in line with Kieran's (2022), when she states that "more studies are needed that research the ways in which students can be assisted in analyzing the visual structure of growing patterns, and in generating related diagrams, so as to better equip them to develop in their functional thinking" (Kieran, 2022, p. 1145). Given that growing patterns are rather limited with respect to functional thinking (Vollrath, 1986), we suggest that the statements should incorporate other mathematical activities designed to treat functions as correspondence (e.g., Kleiner, 1989), for instance, as an 'input-output machine'. The implication raises epistemological questions (e.g., Mason, 2018), such as whether the studies that only look at the first characteristic are enough to inform us on the topic of 'students' functional thinking'. The implications could be that research on functional thinking can only report on a limited understanding, meaning that studies that function as a base for teaching reinforce the discontinuity that was raised by Klein (2016) +100 years ago. Here, we conclude that there is a research gap and a need for more studies considering students' arguments and reasoning when working with functions.

The present paper also had a subordinate aim: to use machine learning and large language models to do FCA. As stated earlier, ontological analysis aims to provide the knowledge body of how a concept is conceptualized through explicit descriptions (Obitko et al., 2004). It informs us how a concept is part of a shared understanding but also tries to reveal patterns. Using only 19 definitions yielded from one search on ERIC–however, a substantial period of 20 years–fulfilled the conditions of data and thereby able to use a cosine similarity matrix (Zhu et al., 2023). It allowed us to test the Hugging Face sentence-transformers library. One limitation with few data sources is that some studies will be excluded, studies that might provide other patterns. Another limitation is that we have chosen to study young students' functional thinking since previous studies tend to focus on older students. This decision means we exclude research on older students, research that can provide another pattern. With this in mind, the patterns that resulted from the analysis were reasonably easy to detect in the cosine similarity matrix. In that way, machine learning was a productive and helpful tool, providing a visual aid that could be used for further qualitative analysis. As such, the results from the present study complement studies like Veith et al. (2023) that look at algebraic thinking, providing insights into how functional thinking is theoretically operationalized. One limitation is that the cosine similarity matrix has no meaning without a qualitative analysis. If more extensive data sets are desired, such additional qualitative analysis would not be probable, given that human input is still needed in these methods. An implication is that given the fast development of machine learning and AI, we argue it is important to note when these tools are helpful and what is still needed to make sense of the outcome.

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REFERENCES

Afonso, D., & McAuliffe, S. (2019). Children's capacity for algebraic thinking in the early grades. *African Journal of Research in Mathematics, Science and Technology Education, 23*(2), 219-232. https://doi.org/10.1080/18117295.2019.1661661

Afriyani, D., Sa'dijah, C., Subanji, S., & Muksar, M. (2018). Characteristics of students' mathematical understanding in solving multiple representation task based on solo taxonomy. *International Electronic Journal of Mathematics Education, 13*(3), 281-287. https://doi.org/10.12973/iejme/3920

Ambrus, G., Filler, A., & Vancso, O. (2018). Functional reasoning and working with functions: Functions/mappings in mathematics teaching tradition in Hungary and Germany. *The Mathematics Enthusiast, 15(*3), 429-454. https://doi.org/10.54870/1551-3440.1439

Beeman, E., Runge, J., Holden, L., Maxwell, D., & Ko, Y. Y. W. (2024). The chain rule does not have to be a pain rule. *The Mathematics Enthusiast, 21*(1), 315-326. https://doi.org/10.54870/1551-3440.1628

Blanton, M. L., & Kaput, J. J. (2011). Functional thinking as a route into algebra in the elementary grades. In J. Cai, & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 5-23). Springer. https://doi.org/10.1007/978-3-642-17735-4_2

Blanton, M., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2017). A progression in first- grade children's thinking about variable and variable notation in functional relationships. *Educational Studies in Mathematics, 95*(2), 181-202. https://doi.org/10.1007/s10649-016-9745-0

Blanton, M., Stephens, A., Knuth, E. J., Gardiner, A. M., Isler, I., & Kim, J.-S. (2015). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for Research in Mathematics Education, 46*(1), 39-87. https://doi.org/10.5951/jresematheduc.46.1.0039

Blomqvist, A., & Sumpter, L. (2024). What is functional thinking? An ontological analysis of different definitions. In *Psychology of Mathematics Education Conference (PME 47)*, Auckland, New Zealand, 17-21 July, 2024 (Vol. 1, pp. 109-109). IGPME.

Burton, L. (1984). Mathematical thinking: The struggle for meaning. *Journal for Research in Mathematics Education, 15*(1), 35-49. https://doi.org/10.5951/jresematheduc.15.1.0035

Carraher, D. W., & Schliemann. A. D. (2007). Early algebra and algebraic reasoning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 669-704). Information Age Publishing.

Chazan, D. (1996). Algebra for all students. *Journal of Mathematical Behavior, 15*(4), 455-477. https://doi.org/10.1016/S0732-3123(96)90030-9

Chimoni, M., Pitta-Pantazi, D., & Christou, C. (2018). Examining early algebraic thinking: insights from empirical data. *Educational Studies in Mathematics, 98*(1), 57-76. https://doi.org/10.1007/s10649-018-9803-x

Chimoni, M., Pitta-Pantazi, D., & Christou, C. (2023). Unfolding algebraic thinking from a cognitive perspective. *Educational Studies in Mathematics, 114*(1), 89-108. https://doi.org/10.1007/s10649-023-10218-z

Ding, R., Huang, R., & Deng, X. (2023). Multiple pathways for developing functional thinking in elementary mathematics textbooks: A case study in China. *Educational Studies in Mathematics, 114*(2), 223-248. https://doi.org/10.1007/s10649-023-10237-w

Georges, J. S. (1946). Teaching functional thinking in mathematics. *School Science and Mathematics, 46*(8), 733-748. https://doi.org/10.1111/j.1949-8594.1946.tb05946.x

González-Polo, R. I., & Castaneda, A. (2024). Progressive mathematics of functions in secondary school students using a free-fall activity. *International Electronic Journal of Mathematics Education, 19*(1), Article em0765. https://doi.org/10.29333/iejme/14108

Häggström, J. (2005). Begreppet funktion i historisk belysning [The concept of function in historical lighting]. *Normat, 53*(2), 82-92.

Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking from childhood to adolescence: An essay on the construction of formal operational structures* (translated by Anne Parsons and Stanley Milgram). Routledge & Kegan Paul Ltd. https://doi.org/10.1037/10034-000

Juter, K. (2012). The validity of students' conceptions of differentiability and continuity. In C. Bergsten, E. Jablonka, & M. Raman (Eds.), *Evaluation and comparison of mathematical achievement: Dimensions and perspectives* (pp. 121-130). Linköping University.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. Kaput, D. W. Carraher, & M. L. Blanton (Eds), *Algebra in the early grades* (pp. 5-17). Routledge. https://doi.org/10.4324/9781315097435-2

Kieran, C. (2022). The multi-dimensionality of early algebraic thinking: Background, overarching dimensions, and new directions. *ZDM–Mathematics Education, 54*(6), 1131-1150. https://doi.org/10.1007/s11858-022-01435-6

Kjeldsen, T. H., & Lützen, J. (2015). Interactions between mathematics and physics: The history of the concept of function–Teaching with and about nature of mathematics. *Science & Education, 24*(5), 543-559. https://doi.org/10.1007/s11191-015-9746-x

Klein, F. (2016). *Elementary mathematics from a higher standpoint volume I: Arithmetic, algebra, analysis*. Springer. https://doi.org/10.1007/978-3-662-49445-5

Kleiner, I. (1989). Evolution of the function concept: A brief survey. *The College Mathematics Journal, 20(*4), 282-300. https://doi.org/10.1080/07468342.1989.11973245

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics, 67*, 255-276. https://doi.org/10.1007/s10649-007-9104-2

Mason, J. (2018). *Qualitative researching* (3rd ed.). SAGE.

Moss, D. L., Boyce, S., & Lamberg, T. (2020). Representations and conceptions of variables in students' early understandings of functions. *International Electronic Journal of Mathematics Education, 15*(2), Article em0564. https://doi.org/10.29333/iejme/6257

Obitko, M., Snasel, V., Smid, J., & Snasel, V. (2004). Ontology design with formal concept analysis. *CLA, 128*(3), 1377-1390.

Pang, J., & Sunwoo, J. (2022). Design of a pattern and correspondence unit to foster functional thinking in an elementary mathematics textbook. *ZDM–Mathematics Education, 54*(6), 1315-1331. https://doi.org/10.1007/s11858-022-01411-0

Pinto, E., Cañadas, M. C., & Moreno, A. (2022). Functional relationships evidenced and representations used by third graders within a functional approach to early algebra. *International Journal of Science and Mathematics Education, 20*(6), 1183-1202. https://doi.org/10.1007/s10763-021-10183-0

Pittalis, M., Pitta-Pantazi, D., & Christou, C. (2020). Young students' functional thinking modes: The relation between recursive patterning, covariational thinking, and correspondence relations. *Journal for Research in Mathematics Education, 51*(5), 631-674. https://doi.org/10.5951/jresematheduc-2020-0164

Pitta-Pantazi, D., Chimoni, M., & Christou, C. (2020). Different types of algebraic thinking: An empirical study focusing on middle school students. *International Journal of Science and Mathematics Education, 18*(5), 965-984. https://doi.org/10.1007/s10763-019-10003-6

Radford, L. (2006). Algebraic thinking and the generalization of patterns: A semiotic perspective. In S. Alatorre, J. Cortina, M. Sáiz, & A. Mendez (Eds.), *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (vol. 1, pp. 2-21). PME-NA.

Ramírez, R., Cañadas, M. C., & Damián, A. (2022). Structures and representations used by 6th graders when working with quadratic functions. *ZDM–Mathematics Education, 54*(6), 1393-1406. https://doi.org/10.1007/s11858-022-01423-w

Sidorov, G., Gelbukh, A., Gómez-Adorno, H., & Pinto, D. (2014). Soft similarity and soft cosine measure: Similarity of features in vector space model. *Computación y Sistemas, 18*(3), 491-504. https://doi.org/10.13053/CyS-18-3-2043

Smith, E. (2008). Representational thinking as a framework for introducing functions in the elementary curriculum. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades*. Lawrence Erlbaum Associates/Taylor & Francis Group and National Council of Teachers of Mathematics.

Somasundram, P., Akmar, S. N., & Eu, L. K. (2019). Pattern generalisation by year five pupils. *International Electronic Journal of Mathematics Education, 14*(2), 353-362. https://doi.org/10.29333/iejme/5719

Sproesser, U., Vogel, M., Dörfler, T., & Eichler, A. (2022). Changing between representations of elementary functions: Students' competencies and differences with a specific perspective on school track and gender. *International Journal of STEM Education, 9*, Article 33. https://doi.org/10.1186/s40594-022-00350-2

Stephens, A. C., Fonger, N., Strachota, S., Isler, I., Blanton, M., Knuth, E., & Murphy Gardiner, A. (2017). A learning progression for elementary students' functional thinking. *Mathematical Thinking and Learning, 19*(3), 143-166. https://doi.org/10.1080/10986065.2017.1328636

Sumpter, L. (2016). Two frameworks for mathematical reasoning at preschool level. In T. Meaney, O. Helenius, M. L. Johansson, T. Lange, & A. Wernberg (Eds.), *Mathematics Education in the Early Years: Results from the POEM2 Conference, 2014* (pp. 157-169). Springer.

Tanışlı, D. (2011). Functional thinking ways in relation to linear function tables of elementary school students. *The Journal of Mathematical Behavior, 30*(3), 206-223. https://doi.org/10.1016/j.jmathb.2011.08.001

Veith, J. M., Beste, M. L., Kindervater, M., Krause, M., Straulino, M., Greinert, F., & Bitzenbauer, P. (2023). Mathematics education research on algebra over the last two decades: Quo vadis? *Frontiers in Education, 8*. https://doi.org/10.3389/feduc.2023.1211920

Vollrath, H-J. (1986). Search strategies as indicators of functional thinking. *Educational Studies in Mathematics, 17*(4), 387-400. https://doi.org/10.1007/BF00311326

Warren, E. A., & Cooper, T. J. (2005). Introducing functional thinking in year 2: A case study of early algebra teaching. *Contemporary Issues in Early Childhood, 6*(2), 150-162. https://doi.org/10.2304/ciec.2005.6.2.5

Warren, E. A., Cooper, T. J., & Lamb, J. T. (2006). Investigating functional thinking in the elementary classroom: Foundations of early algebraic reasoning. *The Journal of Mathematical Behavior, 25*(3), 208-223. https://doi.org/10.1016/j.jmathb.2006.09.006

Warren, E. A., Miller, J., & Cooper, T. J. (2013). Exploring young students' functional thinking. *PNA, 7*(2), 75-84. https://doi.org/10.30827/pna.v7i2.6131

Wilkie, K. J., & Ayalon, M. (2018). Investigating years 7 to 12 students' knowledge of linear relationships through different contexts and representations. *Mathematics Education Research Journal, 30*, 499-523. https://doi.org/10.1007/s13394-018-0236-8

Wilkie, K. J., & Clarke, D. M. (2016). Developing students' functional thinking in algebra through different visualisations of a growing pattern's structure. *Mathematics Education Research Journal, 28*(2), 223-243. https://doi.org/10.1007/s13394-015-0146-y

Xolocotzin, U., Medrano-Moya, A. M., & Rojano, T. (2022). Starting points: Understanding children's pre-instructional intuitions about function tables. *ZDM–Mathematics Education, 54*(6), 1363-1376. https://doi.org/10.1007/s11858-022-01424-9

Zhu, X., Liu, H., & Peng, X. (2023). A study on similarity and difficulty evaluation of elementary school mathematics application problems based on cosine similarity and AHP. *Curriculum and Teaching Methodology, 6*(20), 108-115. https://doi.org/10.23977/curtm.2023.062017