International Electronic Journal of Mathematics Education
2025, 20(2), em0822
e-ISSN: 1306-3030
https://www.iejme.com

MODESTUM

Research Article

OPEN ACCESS

Use of parameters in equations and systems of linear equations: A proposal to boost variational thinking

Luis E. Hernández-Zavala [1*], Claudia Acuña-Soto [1], Vicente Liern [2]

[1] Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional (CINVESTAV), MEXICO
[2] Departamento de Matemáticas para la Economía y la Empresa, Facultad de Economía, Universitat de València, SPAIN
*Corresponding Author: luisenri.hernadez@cinvestav.mx

Citation: Hernández-Zavala, L. E., Acuña-Soto, C., & Liern, S. (2025). Use of parameters in equations and systems of linear equations: A proposal to boost variational thinking. *International Electronic Journal of Mathematics Education, 20*(2), em0822. https://doi.org/10.29333/iejme/16005

ARTICLE INFO

Received: 20 Jun 2024

Accepted: 21 Jan 2025

ABSTRACT

Students often instrumentally use variables and unknowns without considering the variational thinking behind them. Using parameters to modify the coefficients or unknowns in equations or systems of linear equations (without altering their structure) involves consciously incorporating variational thinking into problem-solving. We will test the scope of this approach to undergraduate students by using contextual problems modelled with systems of linear equations that have one solution, infinitely many solutions, or none. In this context, knowing the solution was not enough to decide; instead, modifications to the system were necessary, and incorporating parameters proved to be very useful for this purpose. The goal is for students, in addition to seeing variational thinking as a valuable strategy for determining the validity of a solution, to develop the ability to distinguish between unknowns, variables, and parameters.

Keywords: parameters, variables, unknowns, systems of linear equations, context

INTRODUCTION

The teaching of mathematics should address tasks that involve problems related to data processing and decision-making where parameters are used, not only as part of an operational methodology, but to modify or specify a good portion of the models (Carlsson & Korhonen, 1986). A clear example is found in equations, where it is possible to modify coefficients or unknowns in such a way that they can lead to *valid solutions* to the problems posed (Chinneck, 1997).

Parameters, such as variables of a dual nature, can be used as active variables (they vary when necessary) or inactive variables (they behave as a constant value) depending on the needs (Bardini et al., 2005; Epp, 2011; Freudenthal, 1983). If one focuses on linear equations, the existence or *ad hoc* incorporation of parameters allows intervention on one or more of their elements individually or simultaneously, and this facilitates decision-making (Wendell, 1997). In other words, the notion raised is what should be transformed and to what extent for the solution of the mathematical problem to truly be a decision. Of course, the validity of the solution depends on the context of the problem and the needs of the users (Chiang & Wainwright, 1984; Robbins & Judge, 2017).

If the objective is to solve a System of Linear Equations (SLE, for its acronym) and there is no uncertainty or imprecision in the model, the system has a solution, infinite solutions or no solution (it is said to be incompatible) and by solving it the objective has been achieved. However, when the system represents a real situation, which has surely been simplified to facilitate its handling, a multitude of situations arise in which solving the SLE is not enough; and in these situations, using parameters becomes very useful (Carlsson & Korhonen, 1986; Chinneck, 1997). All numerical values can be affected by inaccuracies or errors that determine the solution and future decision making (Kaufman & Aluja, 1987). However, this is not the most serious situation that one may face. If a real situation that works is modeled it is impossible for there to be no solution, because the real situation does have a solution. In such a case, the introduction of parameters is even more advisable (Dorfman, 1987).

To show the use of parameters as instruments that can *modify* a problem, this paper will progressively address two situations where solutions are a challenge, namely:

(a) when the solutions are infinite, and the parameter allows one to find a functional expression whose domain can be chosen according to the user's needs; and

(b) when the SLE, despite representing a real context, has no solution.

Table 1. Two systems of linear equations without solutions

Original system of equations	Transformed system of equations
$\left.\begin{array}{r}2x+y=4\\6x+3y=6\end{array}\right\}$	$\left.\begin{array}{r}2x+\alpha y=4\\6x+3\alpha y=6\end{array}\right\}\ \alpha\in\mathbb{R}.$

Table 2. Two approaches to the same relationship

Using Variables		Using parameters (as a dual variable)
$2x+3y=20$ $y=\dfrac{20-2x}{3}$	$\begin{array}{c\|c} x & y \\ \hline -2 & 24/3 \\ -1 & 22/3 \\ 0 & 20/3 \\ 1 & 18/3 \\ 2 & 16/3 \end{array}$	$\left.\begin{array}{r}2x+3y=20\\x=\alpha\\y=\dfrac{20-2\alpha}{3}\end{array}\right\},\ \alpha\in\mathbb{R}.$
Exploration with specific values		Continuous exploration of a domain

We shall show that in both situations the use of parameters is very useful but, of course, it does not do away with the difficulties in all cases. In case (a), for the solutions of a system to be expressed as a function of a parameter, the hypotheses of the *implicit function theorem* must be verified. In case (b), adding parameters to a SLE does not guarantee the existence of a solution. For instance, an example is shown in **Table 1** that has no solution (left), and by using a parameter to modify the coefficient of the variable "y", the system still has no solution.

One cognitive effect of parameter treatment, which can broaden the perspective of students, is to introduce them to different modes of variational thinking as can be seen in the simple equation shown in **Table 2**. The thinking that is enhanced when x and y are unknowns raised in the "usual" way (on the left side) is different from that fostered when a parameter offers a functional treatment that highlights the variability relationship (on the right side).

Under the above conditions, in this paper the authors research the way undergraduate students, with no background in dealing with parameters, adopt their use as variables for modifying linear equations or their systems. The modifications are treated with graphic representations to verify and observe the effects they produce on the equations.

To achieve the objective, the authors undertook their inspection using problems designed to analyze the progressive transformation of semiotic categories related to *expression and content* associated with the use of parameters.

Background: System of Linear Equations and the Context

Research on the learning of Systems of Linear Equations reports that our students have difficulty making sense of the algebraic structure of equation systems. This is in part because of operational overuse and also because of their structural peculiarities, which can be interpreted through conceptions that do not match mathematical relationships (Oktaç, 2018), in the case of single or infinite solutions; integer, fractional or real (Larson & Zandieh, 2013; Oktaç & Trigueros, 2010; Smith et al., 2022a; Smith et al., 2022b; Zandieh & Andrews-Larson, 2019) or simply because of the form that solutions take (DeVries & Arnon, 2004; Sfard & Linchevski, 1994).

The results mentioned attest to the complexity of the structural aspects and the conceptions formulated around the resolution of linear equations or their systems, as well as the existence or uniqueness of their solutions (Hernández-Zavala et al., 2023).

These changes become more important when the initial system has no solution, even though the context guarantees that a solution does exist in practice. Making decisions about what elements of the system one wants to change and to what extent can make it possible to arrive at a decision by modifying both the equations and the solutions (Yoneda & Celaschi, 2013). This is one of the reasons that parameters can be transformed into a useful tool in areas such as Economics and Finance, where it is common to find systems without solutions or with an infinite number of them, and where it is common practice to use parameters to find valid solutions and general functional expressions to determine particular solutions (Serrano et al., 2010; Parra & Otero, 2017).

In mathematics education aimed at solving problems, the use of *context,* which could in principle be a useful reference for users, is recommended (Verschaffel et al., 2020). Under this heading, the proposals include formulations of problems associated with physical or social phenomena (Roth, 1996) that include practical (De Corte et al., 2000) or scientific (Sokolowski et al., 2011) situations.

The *context* in mathematics education can also refer to situations that do not copy immediate reality and refer to situations that are *experimentally real,* that is they may come from fictitious contexts, albeit they are proposed based on realistic logic (Clarke & Roche, 2018; Van den Heuvel-Panhuizen, 2005).

REFERENCES FRAMEWORK: SEMIOTICS OF PARAMETERS IN LINEAR EQUATIONS

From the point of view of the semiotics of language, which has been adapted to the case of mathematics, as a dual variable the parameter can be proposed and used as a sign determined by an *expression-content* relationship*,* Hjelmslev (1943) said sign is based on a signifier (expression) and a meaning (content) that are related and constitute a unit called a *sign function* that establishes a particular relationship between the form-expression and form-content that said sign takes (Trabant, 1987). The relationship is also known as a *semiotic function* (Eco, 1972; Godino, 2003; Rondero & Font, 2015). The semiotic function links the *senses* that can be progressively developed through use, and that outline the meanings through the expression/content duo.

Table 3. The same *x expression,* different contents

Mathematical Object	Expression (signifier)	Content (meaning)
In a linear equation with one unknown	$x + 7 = 8$	It represents an unknown the value of which makes the equation true.
In a linear equation with two unknowns[1]	$x + y = 8$	It represents a variable with infinite values that depend on y.
In a system of linear equations with infinite solutions	$\left.\begin{matrix} x + y + z = 10 \\ 2x + 3y + 5z = 25 \end{matrix}\right\}$	It represents a variable that can be transformed into a parameter of a dual nature

In this context, the expression constitutes an equation with two unknowns. It is important to note that the entities involved change ontologically throughout the solution process. Given that the equation possesses infinite solutions, the symbols *x* and *y* are interpreted as variables.

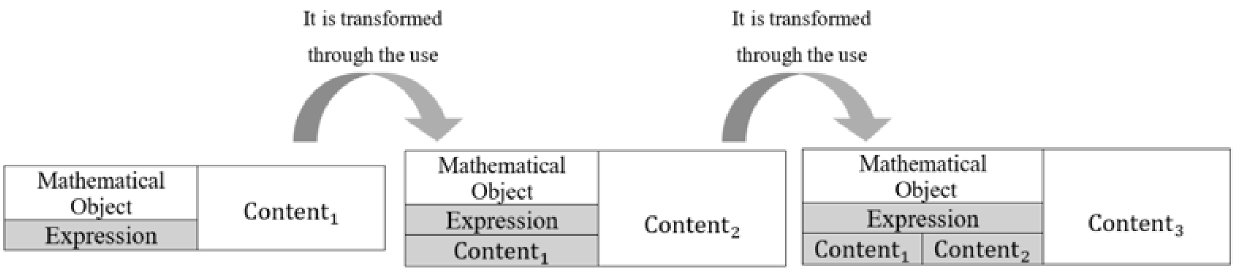

Figure 1. Semiotic articulation - Evolution of expression/content (Adapted from Rondero & Font, 2015)

However, even in the case of unknowns, the same expression (signifier) can be related to different contents (meanings). For instance, let us look at the role of sign *x* in the equations found in **Table 3**.

This phenomenon, in which an expression is assigned more than one distinct content, is related to a *semiotic articulation* (Rondero & Font, 2015) immersed in the *language-game* proposed by Wittgenstein (1953), where the meanings of language are determined by the rules of use assigned to it by the users. In the case at hand, in which the variables are not addressed, it is advisable to associate the expression with the ontological nature of each object and the associated strategies, depending on the uses in question. Thus, we have unknowns as indeterminate elements, whose objective is to be determined through operational processes.

Figure 1 has a diagram depicting the process of transforming the *senses* of the *expression* as a result of the semiotic function by way of its use under different conditions.

As can be seen in **Figure 1**, semiotic articulation occurs with a sequence of progressive meanings associated with the contents that rely on the preceding duo and that are transformed by the use made of the expression under different conditions, which causes the incorporation of new contents (Font & Contreras, 2008, p.8). Thus, the components of *Content* $_1$, *Content* $_2$, ..., are supported by the previous duos by way of *Expression* $_{n-1}$ + *Content* $_n$ relationship, as the basis for a new *Expression* $_n$ and a new *Content* n+1 (Font & Contreras, 2008).

In this iterative process, an expression may be the same as a previous expression, although it will bear in mind the new meaning incorporated through the treatment of the expression. This makes it an enriched sign, unlike with the previous sign (Rojas Garzon, 2015).

In addition, one must consider the complexity of the mathematical object, since "in some circumstances, mathematical objects participate as single entities (which are supposed to be previously known), while in others they come into play as systems that must be broken down for them to be studied" (Rondero & Font, 2015, p. 30). The parameter, understood as a single entity, operates as a variable that assumes a specific value. Whereas its variable nature implies that there is a family of systems that modify the original system, so that at the same time their variation provides a family of associated systems of linear equations, which rely on the parameter where each SLE has its respective solution.

METHODOLOGY

The general objective of this study was to test the didactic proposal that parameters are dual variables capable of modifying conditions in systems of linear equations in two situations: when SLE have no solution or when they have infinite solutions. Our object of study is the progressive development of the *expression/content* relationship among university students on operation of the parameter as a variable to modify SLE in the situations mentioned above.

In this research we use a qualitative interpretative methodology, based on a proposal for didactic exploration. This analytical method aims to enrich didactic decisions about the use of parameters; improve instruction; and test mathematical learning models for students (Cobb & Gravemeijer, 2014; Lesh & Kelly 2000; Steffe & Thompson 2000; Steffe & Ulrich, 2020).

This methodology offers researchers the opportunity to analyze subjects' progress through mathematical communications and to bridge the gap between teaching and research, as well as theory and practice. Its primary focus is to understand the impact of teaching approaches on the reasoning and level of mathematical knowledge of the subjects (Cobb, 2000; Czarnocha & Prabhu, 2004; Lesh & Kelly, 2000; Steffe & Thompson, 2000). This methodology makes it possible to analyze students' progress through the study of various instructional episodes aimed at clarifying their understanding of mathematical concepts and operations, as well

Table 4. Organization of the *exploration*

	Session 1	Session 2	Session 3
Theme (s)	Introduction Unknown Variables	Introduction to Parameters	Parameters in contextual situations
Activit(y/ies)	Activity 1 Activity 2	Activity 3	Activity 4

Isaías compró dos kilogramos de uvas y tres de manzanas y gastó 78 pesos. José pagó un total de 132 pesos por cinco kilogramos de uvas y cuatro de manzanas. Es posible determinar el precio de cada producto resolviendo el siguiente sistema de ecuaciones lineales: $\left.\begin{matrix} 2x+3y=78 \\ 5x+4y=132 \end{matrix}\right\}$ La incógnita x representa el número desconocido de pesos que cuesta el kilogramo de uvas y la incógnita y representa el número desconocido de pesos que cuesta el kilogramo de manzanas. *x=precio de 1 kg. de uvas,* *y=precio de 1 kg. de manzanas.* (Original Spanish text)	Isaías bought two kilograms of grapes and three kilograms of apples and spent 78 pesos. José paid a total of 132 pesos for five kilograms of grapes and four kilograms of apples. It is possible to determine the price of each product by solving the following system of linear equations: $\left.\begin{matrix} 2x+3y=78 \\ 5x+4y=132 \end{matrix}\right\}$ The unknown x represents the unknown number of pesos that a kilogram of grapes costs; and the unknown y represents the unknown number of pesos that a kilogram of apples costs. *x = price of 1 kg of grapes,* *y = price of 1 kg of apples.*

Figure 2. The unknown in an SLE (Source: Authors' own elaboration)

as the transformations that take place in these domains (Swan, 2020). In the context of this research, it is particularly useful to observe the possible development of variational thinking throughout the different instructional episodes framed in the didactic exploration.

The means used were based on the introduction of the structural properties of unknowns, variables and parameters -the latter as dual-natured variables- to pose a series of tasks in a questionnaire solved in person. During the activities some systems of linear equations were modified using parameters to observe their effects in algebraic, graphic and contextual environments. That is to say, the experiment intends to show the potential of parameters as modifiers of system solutions.

The intervention process through parameters was based on the graphic management of five applets of the *GeoGebra Classroom* virtual format, which provided dynamic resources to modify the graphs of lines based on the change in the value of the parameters, which produced families of lines associated with specific solutions. At the same time the contextual problems enabled making sense of the components of the problem (products; prices; earnings) and their dependency relationships within the equation. Our point of departure was fixed earnings, from which we then perceived the change in the values of the parameter once its variability was established, as well as the effect that this change had on the solutions of the new SLEs, to finally find the solutions that are valid for the user (decision-making).

An iterative exploration method was developed that consisted of:

(1) Solve the problems posed;

(2) Discuss the solutions; and

(3) Promote a collective question/answer and comment activity on the new suggestions, to then return to point (1) and solve the new problems to repeat the cycle.

The sample was made up of 21 students from an Algebra I course. The students had no prior knowledge of the use of parameters. In this study, the authors will report on three sessions of the exploration, with each session lasting an average of 50 minutes.

Data was obtained from written responses and collective discussions among students and student researchers concerning the activities described in **Table 4**.

Session 1

In this session, students were presented with the *unknown* as a (temporarily) indeterminate quantity that can be calculated with the information given, and the *variable* as an *indeterminate quantity* that varies and describes general relationships. Two examples were addressed, emphasizing the role played by each entity despite having the same expression, as shown in **Figure 2**.

This example shows the nature of the unknowns (x and y) as representing unknown objects, but which are already determined by the conditions expressed in the equation. In this case x and y represent specific quantities (weights per kg of grapes or apples) associated with the problem.

La tarifa base de un taxi en la Ciudad de México es de 13.10 pesos y se añade 5.2 pesos cada minuto después de iniciado el viaje. Con estas cantidades podemos establecer una expresión general que representa el precio de viajar en taxi durante un indeterminado lapso de tiempo: $y = 13.10 + 5.2x,$ o bien, $f(x) = 13.10 + 5.2x,$ donde x representa el tiempo de viaje en minutos e y representa el precio en pesos. Esta expresión mide el precio del viaje al variar el número de minutos. (Original Spanish text)	The base fare for a taxi in Mexico City is 13.10 pesos and 5.2 pesos are added for every minute after the start of the trip. With these quantities one can establish a general expression that represents the price of travel by taxi for an indefinite period of time: $y = 13.10 + 5.2x,$ or $f(x) = 13.10 + 5.2x,$ where x represents the travel time in minutes and y represents the price in pesos. This expression measures the price of the trip by varying the number of minutes.

Figure 3. Role of the variable in a situation of variation (Source: Authors' own elaboration)

Table 5. Structure of Session 1

		Activity	Teaching and learning objectives
Session 1	Unknowns	**A1:** Solve three tasks related to linear equations with one and two unknowns.	- Interpret the unknown as an entity that represents an *indeterminate* and *unknown* quantity that must be determined. - Make sense of the solutions (contextual, algebraic and graphic) and identify processes when operating with the unknown. - Interpret the sign-context relationship (unknown-context) and the units it represents in the extra-mathematical environment.
	Variables	**A2**: Solve 2 tasks that make sense of the proposed situations.	- Give meaning to the variable as an entity that represents *indeterminate* quantities that vary and enable describing and predicting phenomena analytically. - Interpret the solutions (contextual, algebraic and graphic) and identify the processes when operating with the variable. - Interpret the sign-context relationship (variable-context) and the units in the extra-mathematical environment.

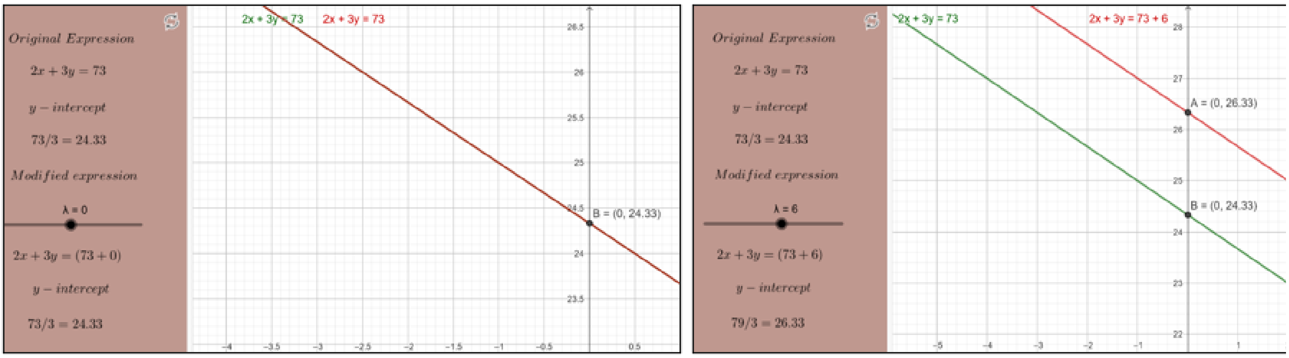

Figure 4. Role of the parameter as a modifier (Source: Authors' own elaboration)

The role of *x* and *y* as variables was also shown, which in this case explicitly represent variational thinking (see **Figure 3**).

The specific teaching and learning objectives of Session 1 are shown in **Table 5**.

Session 2

In this session, we initially presented the parameters with the usual Greek letter nomenclature, although it was noted that the choice was optional, and students could use other types of letters.

We tested the effects of modifying the coefficients and the independent term of linear equations, depending on user needs. As shown in the following example (**Figure 4**), where the parameter modifies the independent term of a linear equation with infinite solutions, to analyze its effect we first verified it algebraically and then we observed the effect of the modifications on the *ordinate at origin* (y-intercept) and the *slope* of the graphs of the respective lines.

In Section 2, the activities described in **Table 6** were also performed.

The objectives of Session 2 are described in **Table 7**.

Table 6. Modifications addressed in Session 2

Task	Type of transformation
Entering a parameter into a linear equation with one unknown.	Modify the unknown with a parameter to obtain a family of linear equations.
Entering a parameter into a rectangular (*2x3*) SLE.	Transform a variable into a parameter to modify the algebraic expression and solutions by choosing the domain.
Entering at least one parameter as the sum of the coefficient, in the form (a+λ)	Modify the coefficient of a term to alter the slope of the associated line and obtain a SLE family with solutions.

Table 7. Structure of Session 2

	Activity	Teaching and learning objectives
Session 2 Parameters	**A3**: Solve 5 tasks that involve the algebraic and geometric interpretation of some modifications to linear equations with the introduction of parameters.	- Introduce a new algebraic entity: the parameter, as a dual-natured variable. - Identify the type of solution depending on the modified element when operating with the parameter, as well as interpret its functions and properties. - Observe the algebraic and geometric effects to enrich the sense and observe the change caused by the parameters in the SLEs.

El señor de la tienda de la esquina introdujo dos productos nuevos: cacao y café. El último mes obtuvo una ganancia total de 6800 pesos por la venta de esos dos productos. El kg. de cacao lo vende por 120 pesos y el kg. de café por 40 pesos. Recientemente abrió una nueva tienda en otra zona de la ciudad con mayor poder económico, por lo que en ésta ha decidido incrementar los precios del cacao y del café a 180 y 60 pesos el kg., respectivamente. Espera obtener una ganancia de 8000 pesos con la misma cantidad de kg. vendidos que en la primera tienda.

Con los datos proporcionados, podemos definir dos incógnitas:

x=kg. de cacao, y= kg. de café.

De esta manera, podemos establecer el siguiente sistema de 2 ecuaciones lineales con 2 incógnitas:

$$\left.\begin{array}{l}120x + 40y = 6800\\180x + 60y = 8000\end{array}\right\}$$

¿Será posible vender la misma cantidad de kg. de cada producto en ambas tiendas para obtener las ganancias deseadas?

(Original Spanish text)

The man from the corner store brought in two new products: cocoa and coffee. Last month, he had total earnings of 6800 pesos from the sale of these two products. He sells the kg of cocoa for 120 pesos and the kg of coffee for 40 pesos. He recently opened a new store in another area of the city that has a higher purchasing power, so he decided to increase the prices of cocoa and coffee to 180 and 60 pesos per kg, respectively, in that area. He expects to make a profit of 8000 pesos from the same number of kg sold as in the first store.

With the data provided, two unknowns can be defined:

x = kg of cocoa, y = kg of coffee.

Hence the following system of 2 linear equations with 2 unknowns can be established:

$$\left.\begin{array}{l}120x + 40y = 6800\\180x + 60y = 8000\end{array}\right\}$$

Will it be possible to sell the same kg amount of each product in both stores to obtain the desired earnings?

Figure 5. Contextual situation (incompatible SLE) (Source: Authors' own elaboration)

Session 3

This session involved four experimentally real problems that included contextual references that the students modified to obtain a valid solution -from an infinite set of solutions (from a SLE family)- that would solve the problem. For example, in the next task (**Figure 3**) students are asked to modify at least one element of the system (incompatible) to obtain a solution. In this task, 5 *applets* were proposed to simulate changes in each original system parameter to verify the effect on the solution.

In terms of the context, this translated into modifying the prices of the products involved (the numerical coefficients of the unknowns) the price of cocoa or the price of coffee, which affects the kg amount of each product that can be purchased; if the profits are fixed (constant terms) and we require valid solutions, the problem becomes one of decision-making.

In the activity proposed in **Figure 5**, the authors found it necessary to highlight a logical sequence that is often overlooked in the classroom. In general, the student is left with the impression that ∃ Mathematical Solution ⇒ ∃ Solution to the real problem. However, in the problem used in Session 3 (see **Figure 5**), the implication given is specifically ∄ Mathematical Solution ⇒ ∃ Solution to the real problem. The fact that SLE has no solution answers the question: Will it be possible to sell the same kg of each product in both stores to obtain the desired profits? Clearly, the seller's wishes are not possible.

Table 8. Structure Session 3

	Activity		Teaching and learning objectives
Session 3	Parameters in contextual situations	**A4:** Four contextual situations were proposed that involve introducing parameters to modify and make decisions based on valid situations.	- Make sense of the parameter as a tool that can modify SLEs to provide valid solutions for decision-making. - Identify the uses of the parameter to modify them. - Graphically verify the validity of solutions for decision-making. - Make sense of the sign-context relationship (parameter-context) and the units it represents in the extra-mathematical environment.

Table 9. Answers to Question 1 and 2

In the equation x+y $= 30$			
Questions	Number of students per answer		
How many solutions are there?	Infinite 16		Other 5
How would you represent the solutions?	With variables 5	With unknowns 4	Other 5

Table 10. Expression and Content of Session 1

Expression	Contents
x =12 kg of grapes y =18 kg of apples	- Unknown - x, y refers to quantities that are determined by solving the equation or system

The expected objectives of Session 3 are shown in **Table 8**.

In the next section, the authors analyze the results obtained from the activities and the students' answers to several additional questions.

ANALYSIS AND RESULTS

In this *exploration* we will show the general answers and the essence of the dialogues of five students (A1, A2, A3, A4 and A5) who led the discussion in the classroom and considered, together with the researcher (I) the effects of the intervention and the potential for using parameters on the conditions of the SLEs and their solutions in the cases mentioned.

The data analysis was performed using a qualitative method known as *discourse analysis*. This approach enables observing and interpreting conversations between two or more individuals within a particular context. It investigates not only the content of the dialogue but also its expression, the underlying motivations that drive the conversation, and the various types of interactions, sequences, contexts, and structures that define these exchanges (Cohen et al., 2018, p. 688). Some of the dialogues considered relevant are included, and for this the task that gave rise to the reflection is provided. Additionally, all students' responses to the task questions in the *Geogebra Classroom* were considered for the data analysis.

Session 1

In this session students are faced with two situations that include information about the nature of unknowns and variables, as well as elements that show that they are different entities and have different purposes. This information was new to them because it is not part of the current curricular content.

Activity 1 addressed the use and properties of the unknown in linear equations and SLE, as a temporary indeterminate entity that will be calculated in three areas. As an example, we show the treatment of the following contextual problem, where the ordered pair *(x, y)* was determined where x and y represent a specific number of kilograms of grapes and apples, as shown in **Table 9**.

Below, we transcribe part of the dialogue that took place in this session amongst students A1 and A2 with the researcher.

L1. I: What happens with this type of equation [$x + x = 30$]. Can anyone tell me how many solutions this equation has?

L2. A2: Two, it can have two, two... several... several

L3. **I**: How many? 10, 100, a million?

L4. A2: Aha, yes, about 100 about 100, yes

L5. A1: More, isn't it? It's like... it has infinite solutions.

(Dialogue 1, 2023).

The differences between unknown and variable, at this time, do not seem to have repercussions on some of the students interviewed. Only 5 of the 21 students note that to represent and make sense of the expression with infinite solutions it is necessary to use variables instead of unknowns. This is reflected in the following dialogue between researcher **I** and students **A1** and **A2.**

Table 11. Semiotic function between expression and content of Session 2

Expression	Contents
Sign in equation type $ax = b$, as in $3\lambda x = 21$	- Variable that comes into play $-\lambda$ is an object that comes into play with controlled changes to another object.

Regarding the expression/content relationship associated with these two entities (**Table 10**), both equations and SLE, we observed that the "*x*" shares two contents (such as unknown or variable) of the same expression. This is the case because students still do not make sense of the difference between unknown and variable, which causes a weak understanding of equations with infinite solutions.

One conflict that we identified in this introduction is that variability- here taking the form of establishing a general analytical expression- is confused with the number of solutions allowed by the equation, which is a commonly used rhetorical resource in class to characterize the variable.

The difference between unknowns as *undetermined* and *unknown* quantities that can be determined and variables as *undetermined* quantities that analytically predict phenomena is not yet considered in the dialogue.

Session 2

The parameter is introduced as a special dual-natured variable, as an active and inactive variable -or as a variable and constant- that are used to modify equations. In this session, we use the parameter to change some elements of the equations and SLEs and interpret their effects.

Activity 3 of that session proposed to modify a linear equation with an unknown to have a general expression of the relationship, so that the parameter affected the coefficient that it could measure in a range of variation, as shown below:

If the parameter λ is a real number between 0 and 5 ($\lambda \in \mathbb{R}$ such that $0 < \lambda < 5$), then $3\lambda x = 21$ means that $x = \frac{21}{3\lambda},\ 0 < \lambda < 5$.

Unlike the previous activity, in this case the students were able to observe the changes caused by the parameter by choosing values for the parameter. During the discussion, this was the dialogue that ensued between the researcher and student A2:

L9.I: Yes, depending on how you define the literal it will be a parameter, an unknown or a variable.

L10.A2: And are any other [signs] used? For example, if you use theta or beta or alpha, that is, what is it like?

L11.I: It's the same. What you call it doesn't matter. Here I could have called this zeta alpha, etc.

L12.A2: Oh right, [the sign] doesn't matter, what matters is the value added [assigned] to it. Aha!

(Dialogue 2, 2023).

In this activity, students begin to recognize the parameter as an object rather than an unknown, because they have seen how it comes into play to obtain a general expression for the solutions and thus enables them to calculate the solutions. A2 advances the change of sense associated with the parameter in L12 when s/he states that "[the sign] doesn't matter, *what matters is the value added* [assigned] *to it. Aha!*".

At this point one can assert that the sign is associated with the expression that acquires a new different *sense* and that it affects the unknown. Although they have not yet been able to see that the parameter allows them to obtain a family of equations, all of which are potentially useful for calculation purposes.

In terms of the semiotic function, the *expression/content* relationship can be summarized as shown in **Table 11**.

Another task related to a two-variable equation (with infinite solutions) where one of the equations came into play with a parameter to show the change in the equation and its solutions, as shown in the following activity:

Given the following algebraic expression, $2x + 3y = 64$ we can analyze it functionally as $f(x) = \frac{64-2x}{3}$ or parametrically as follows: Let $\lambda \in \mathbb{R}$ such that $x = \lambda$. So, we have the following relationship:$2\lambda + 3y = 64$. Since λ is an active and inactive variable, it can be used here in its inactive form: Given that $\lambda \in \mathbb{R} \Rightarrow 3y = 64 - 2\lambda \Rightarrow y = \frac{64-2\lambda}{3}$. Values can be assigned to λ in each domain so that we obtain certain types of solutions. For example, if $\lambda \leq 32$ that is, $\lambda \in (-\infty, 32]$ only positive values of *y* will be obtained.

During the activity, this was the dialogue that took place between A3 and A2:

L17.A3: Oh yeah, there can be more... [values]

L18.A2: Ah! ... so, does it vary?

L19.I: Yes, the range of variation will depend on the problem and your decision. In principle it is limited to real numbers, but you decide...

L20.A2: Ah! Ok... I get it!

(Dialogue 3, 2023).

Figure 6. Applet - Modifying a Coefficient (Source: Compiled by the authors)

Table 12. Table at the bottom of a column

In the equation $2x + 3y = 60$, is it possible to enter more than one parameter to modify it? Explain your answer	
Responses	*Number of students*
Yes	
Entering at least two parameters by modifying the equation	17
No	
The equation was already defined	4

The condition of the equation, when accepting several values, is considered by A2 to be the property of *varying* at L18 and although this interpretation can be improved -since the variability is not restricted to the change of values, but to the ability to be a general representation- in this case the expression is adequate to accept the availability of different solutions.

Then s/he proposed to modify the coefficient of the unknown *x* in an equation so that the effects of the intervention could be seen in the graphic representation, as shown on **Figure 6.**

This task resulted in the following dialogue between the researcher and students A1 and A3:

L21.A1: Is active [variable] one that we set a value for?

L22.I: Yes, and that may vary... it may be changing

L23.A3: And the inactive [variable] is one that states that it's fixed... it's fixed...

L24.I: Exactly...

L25.I: So, since here is lambda and we can make lambda vary because it's going to be changing, then the slope is going to be changing...

L26.A3: And there it would be active... it would be [an] active variable.

(Dialogue 5, 2023).

In this intervention, a more profound effect was detected on the sense assigned to the parameter when students were asked to find different solutions to the equation or system of equations, which begins to promote the idea that it was possible to:

1. Use it as a variable that controllably affects some of the terms in the equation, and

2. Modify the solutions using the value chosen.

The answers begin to consider the change in the conditions associated with the coefficients of the unknown or the independent term. At this stage, the following results were shown in **Table 12**.

In general, we observed that students have accepted the parameter as a special entity and that it can modify the coefficients of unknowns or constants.

As for the content and expression of algebraic entities, the parameter has been established as a particular algebraic entity, with certain properties and uses. In terms of the *expression/content* relationship, the idea of active and inactive variables is addressed.

In terms of the relationship between expression and content of Session 2, the form it takes can be seen in **Table 13**.

Table 13. Expression and content of λ in Session 2

Expression	Content 1	Expression 1	Content 2
Sign λ in an equation of the type $2\lambda + 3y = 64$ $\Rightarrow y = \dfrac{64 - 2\lambda}{3}$	λ it is a(n) (inactive) variable that allows specific values to be taken	Sign λ in an equation of the type	λ It is a(n) (active) variable that acts on other terms and affects them
	λ It is a parameter	$(a+\lambda)x + by = c$	λ It is a parameter

Session 3

Some experimentally realistic situations were addressed in Session 3. They are represented by linear equations and SLE, starting from the possibility of modifying the systems and where a valid solution (from an infinite set) had to be chosen to solve the situation. The following approach is an example:

We have a budget of 330 pesos to buy coffee and sugar for a month. And we know that a kilogram of coffee costs 90 pesos and a kilogram of sugar costs 30 pesos. How can we know how many kilograms of each product we can buy with that budget?

To resolve this situation and decide on how many kilograms to buy of each product, we will enter a parameter as follows: *Let* $\lambda \in \mathbb{R}$ such that $y = \lambda \Rightarrow 90x + 30\lambda = 330 \Rightarrow 90x = 330 - 30\lambda$in such a way that we now have control over expression.

Now let us suppose that we need to buy the same products, but now we want to use part of the total budget for public transport, taxis or other services. In this case you decide how much money you will set aside and what it will be for. The mathematical expression that represents this situation is as follows:

$$90x + 30y = 330 - \lambda;\ \lambda \in \mathbb{R} \tag{1}$$

Overall, 19 students chose a valid (positive) solution, and based on that situation A1 and A3 had the following discussion:

L48.A1: Professor, so that equation [the expression that represents the solution set] is giving us the... What exactly is that equation giving us? That's giving us coordinates on the Cartesian plane, but that translates into... Sugar and coffee?

L49.I: Yes, this will translate into how many kilos you can buy of sugar and of coffee

L50.A1: Ahh! [student is surprised]

L51.A3: Professor, but that would be solving with the parameter, right?

L52.I: Yes, with the parameter, which gives us lambda. It allows us to have control of the expression and decide how many kilos you want to buy of each product.

L53.A3: So, you decide...

L54.I: Exactly

L55.I: What [the introduction of the parameter] itself allows is to have control of that expression...

L56.A3: What if they can do it differently? If you want to buy two kilos of coffee and four kilos of sugar?

L57.I: Yes, if you can, it's a solution to the equation. If you want to buy 2 kg of coffee... 2.1 kg of sugar, you should buy almost 3 kg of coffee.

L58.A3: Oh, but depending on the money you already had...

L59.I: Exactly, from your 330-peso budget.

L60.A1: Can the slope be calculated with that equation? The slope of all possible solutions?

L61.I: Yes, it can be done [calculates the slope].

(Dialogue 9, 2023).

Here we observe that students have managed to make sense of the parameter that comes into play in linear equations and their systems as a tool that modifies the SLE. It also enables the finding of valid solutions and decision-making associated with the problem posed. Because the equation was modified, the changes made sense in terms of the context.

During the process A1 sets some mathematical relationships: on the one hand, the connection between variables, unknowns and parameters; on the other, s/he realizes that the parameter in its general graphic-dynamic aspect represents a family of objects. S/He expresses the latter respect when s/he says:

Table 14. Expression and content of the parameter λ in Session 3

Expression $_2$	**Content** $_3$	**Expression** $_1$	**Content** $_2$
Sign λ in an equation of the type $(a+\lambda)x+by=c$	λ is a parameter (active and inactive variable) that modifies coefficients (changes them) and is defined in a range.	λ is a parameter of $(90+\lambda)x+30y=300$ or $90x+(30+\lambda)y=300$ or $90x+30y=300+\lambda$	λ modifies the solutions of a family of associated lines, varying the slope and/or the ordinate at the origin (y-intercept).

Table 15. Table at the bottom of a column

Did they use the parameter to arrive at a favorable decision?	
Responses	Number of students
Yes, they used the parameter to choose a valid solution that excludes negatives and identifies an appropriate range where the parameter can vary.	18
No	3

"The slope of all possible solutions" at L48, in addition to the contextual relationship between coordinates and sugar and coffee. One can also see the dependence between the quantity of the weight of the products under the condition of having allocated a fixed budget, by A3 at L56 and L58.

The content associated with the expression developed by the students evolved and at this point they have already redefined that unknowns, variables and parameters are related objects, and that parameters are special variables that change the conditions and solutions of the SLEs (see **Table 14**).

At this stage, the changes in the equation were applied to both variables and to the independent term. In all cases their effects were graphically verified and the difference between unknown and variable was observed as a product of the properties of the parameter.

At the end of the session, the question was addressed. The answers are found in **Table 15**.

DISCUSSION AND CONCLUSIONS

Mathematics education should foster thoughtful reflection and meaningful discussion regarding systems of linear equations. Currently, attention and efforts continue to be focused on calculation methods and algorithms for resolution (Oktaç, 2018; Zandieh & Andrews-Larson, 2019), to the detriment of many other useful tools such as modeling and decision-making. We think that examples such as the one set out in our proposal -a system that does not provide a solution effectively illustrates a real-world scenario that necessitates a decision-making process- can help to provide evidence of shortcomings in our students.

The existing literature on Systems of Linear Equations (SLE) and their interpretation is somewhat limited; however, we align with the perspectives articulated by Smith et al. (2022a, 2022b) regarding the significance of incorporating non-standard SLE-specifically, both incompatible and compatible indeterminate equations-into the classroom curriculum. Such integration can be facilitated through the implementation of *experimentally realistic problems*, which not only cultivate a meaning-rich educational environment but also serve as an instrumental medium for the development of more formal mathematical concepts and skills (Kaiser, 2017; Mevarech et al., 2018; Mevarech & Kramarski, 2014; Van den Heuvel-Panhuizen, 2001).

The experiment performed suggests that the differences between unknowns, variables and parameters are unclear when there is no environment - preferably contextual- that makes it possible for students to make sense of the elements initially involved in the equations (such as prices, kilograms of product, etc.) and then in the relationships established among them. This seems to indicate that students have not been receiving sufficient instruction on the language games involved in systems of linear equations. These findings are consistent with those reported by Zandieh and Andrews-Larson (2019), who note that the symbolization employed by students underwent significant transformations, mainly through renaming variables, introducing new variables and developing reasoning around the concept of parameter.

In this work, we have shown the potential and versatility of the parameter as a variable that, when permitted, can modify or specify mathematical models. It can even be used to analyze the validity of the solutions to the problems posed based on their context. In this context, and in line with various conceptualizations of the parameter, we propose to view this entity as a variable with fixed and variable characteristics (Bardini et al., 2005; Bloedy-Vinner, 1994; Freudenthal, 1983; Keene, 2007). Additionally, we introduce a new aspect by decontextualizing its use. This approach not only allows for the variation of existing variable elements (Drijvers, 2001) but also enables us to modify elements that are typically considered constant, such as the coefficients of an equation. To that end, we proposed not only a method to use these entities to solve linear equations and systems with infinite solutions, but we also provided graphic and contextual meanings that facilitated their use and interpretation.

Graphical representations that allowed manipulating parameter variations played a crucial role in outlining the meanings associated with changes and their effects on systems of linear equations. These representations are particularly valuable in the initial stages of solving *experimentally realistic problems*, as they facilitate the construction of situational and mathematical models by providing an additional source of relevant information (Verschaffel & De Corte, 2016).

In this context, the use of dynamic geometry environments was particularly significant, playing a central role in interpreting the role of parameters in SLEs and their solutions (Turgut & Drijvers, 2021). Such environments allowed students to manipulate the coefficients of the system, explore various solution cases, formulate conjectures and validate them mathematically and contextually (Gol Tabaghi, 2014; Gol Tabaghi & Sinclair, 2013).

These findings underscore the importance of further research on the role of dynamic geometry environments in the teaching and learning of linear algebra, highlighting their potential to enrich conceptual and practical understanding in this area (Aytekin & Kiymaz, 2019; Turgut, 2019; Turgut & Drijvers, 2021; Turgut et al., 2022).

Additionally, in this research we have shown that introducing parameters into a decision-making environment, supported by an environment of dynamic geometry, proved to be a didactically useful resource for teaching them. The foregoing emphasizes the importance of including this entity in textbooks and study programs explicitly and as an object with properties other than those of unknowns and variables.

Author contributions: LHZ: administration of data gathering, conceptualization, formal analysis, investigation, methodology, resources, writing–original draft; **CA:** conceptualization, formal analysis, methodology, writing–reviewing and editing, supervision; **VL:** conceptualization, formal analysis, writing–reviewing and editing; All authors have agreed with the results and conclusions. All authors have agreed with the results and conclusions.

Funding: No funding source is reported for this study.

Ethical statement: The authors stated that, in accordance with local legislation and state universities requirements, ethics review approval was not required for this study. The study did not require the collection of sensitive data. The data used were obtained from public sources or anonymized, complying with local regulations and university requirements. The authors informed the university authorities at the university where the research was conducted of the details of the study. Written informed consent was obtained from the participants prior to conducting the research. All participants were adults and were informed about the research, anonymity, and their right to withdraw consent without adverse repercussions.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

REFERENCES

Aytekin, C., & Kiymaz, Y. (2019). Teaching linear algebra supported by GeoGebra visualization environment. *Acta Didactica Napocensia*, *12*(2), 75-96. https://doi.org/10.24193/adn.12.2.7

Bardini, C., Radford, L., & Sabena, C. (2005). Struggling with variables, parameters, and indeterminate objects or how to go insane in mathematics. In H. L. Chick, & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, No. 17, pp. 129-136).

Bloedy-Vinner, H. (1994). An algebraic mode of thinking - the case of parameter. In P. da Ponte, & J. F. Matos (Eds.*), International Group for the Psychology of Mathematics Education*, Lisbon University (Vol. 2, p. 88-95).

Carlsson, C., & Korhonen, P. (1986). A parametric approach to fuzzy linear programming. *Fuzzy sets and systems*, *20*(1), 17-30. https://doi.org/10.1016/S0165-0114(86)80028-8

Chiang, A. C., & Wainwright, K. (1984). *Instructors manual to accompany fundamental methods of mathematical economics*. (4a ed.). McGraw-Hill Professional.

Chinneck, J. W. (1997). Feasibility and viability. In *Advances in Sensitivity Analysis and Parametric Programming* (pp. 491-531). Springer. https://doi.org/10.1007/978-1-4615-6103-3_14

Clarke, D., & Roche, A. (2018). Using contextualized tasks to engage students in meaningful and worthwhile mathematics learning. *The Journal of Mathematical Behavior*, *51*, 95-108. https://doi.org/10.1016/j.jmathb.2017.11.006

Cobb, P. (2000). Conducting teaching experiment in collaboration with teachers. In E. Kelly, & A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307-333). Lawrence Erlbaum Associates, Publishers.

Cobb, P., & Gravemeijer, K. (2014). Experimenting to support and understand learning processes. In: *Handbook of design research methods in education* (pp. 86-113). Routledge. https://doi.org/10.4324/9781315759593-14

Cohen, L., Manion, L., & Morrison, K. (2018). In L. Cohen, L. Manion, & K. Morrison (Eds). *Research Methods in Education*. Routledge. https://doi.org/10.4324/9781315456539

Czarnocha, B., & Prabhu, V. (2004). Teaching-research and design experiment-two methodologies of integrating research and classroom practice. In R. Giere, A. Karmiloff-Smith, S. Kosslyn, A. Kumar, K. Kumar, & N. Mukunda (Eds.), *E-proceedings of EPISTEME-1, International Conference to Review Research on Science, Technology and Mathematics Education* (pp. 78-80). Goa, India.

De Corte, E., Verschaffel, L., & Greer, B. (2000). Connecting mathematics problem solving to the real world. In *Proceedings of the International Conference on Mathematics Education into the 21st Century: Mathematics for living* (pp. 66-73).

DeVries, D., & Arnon, I. (2004). Solution-What does it mean? Helping linear algebra students develop the concept while improving research tools. *International Group for the Pyschology of Mathematics Education.*

Dorfman, R., Samuelson, P. A., & Solow, R. M. (1987). *Linear programming and economic analysis*. Courier Corporation.

Drijvers, P. (2001). The concept of parameter in a computer algebra environment. In R. Speiser, C. Maher, & C. Walter (Eds), *Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 2-385).

Eco, U. (1972). *EinJiihrung in die Semiotik* [Introduction to Semiotics]. Munich.

Epp, S. (2011). Variables in mathematics education. In Blackburn, Van Ditmarsch, & Soler-Toscano (Eds.), *Tools for teaching logic: Third International Congress, TICTTL* (pp. 54-61). Springer. https://doi.org/10.1007/978-3-642-21350-2_7

Font, V., & Contreras, Á. (2008). The problem of the particular and its relation to the general in mathematics education. *Educational Studies in Mathematics*, *69*(1), 33-52. https://doi.org/10.1007/s10649-008-9123-7

Freudenthal, H. (1983). Didactical phenomenology of mathematical structures. *Educational Studies in Mathematics, 16*(1), 461-491.

Godino, J. D. (2003). *Teoría de las funciones semióticas: Un enfoque ontológicosemiótico de la cognición e instrucción matemática* [Theory of semiotic functions: An ontologico-semiotic approach to cognition and mathematics instruction]. University of Granada.

Gol Tabaghi, S. (2014). How dragging changes students' awareness: Developing meanings for eigenvector and eigenvalue. *Canadian Journal of Science Mathematics and Technology Education*, *14*(3), 223-237. https://doi.org/10.1080/14926156.2014.935528

Gol Tabaghi, S., & Sinclair, N. (2013). Using dynamic geometry software to explore eigenvectors: The emergence of dynamic-synthetic-geometric thinking. *Technology Knowledge and Learning*, *18*(3), 149-164. https://doi.org/10.1007/s10758-013-9206-0

Hernández-Zavala, L., Acuña-Soto, C., & Liern, V. (2023). Los parámetros y las infinitas soluciones en sistemas de ecuaciones lineales [Parameters and infinite solutions in systems of linear equations]. *Bolema: Boletim de Educação Matemática*, *37*, 872-899. https://doi.org/10.1590/1980-4415v37n76a23

Hjelmslev, L. (1943). Prolegomena to a theory of language. *International Journal of American Linguistics*, 19(1). Indiana University.

Kaiser, G. (2017). The teaching and learning of mathematical modeling. In Cai. J (Ed.), *Compendium for research in mathematics education* (pp. 267-291). National Council of Teachers of Mathematics.

Kaufman, A., & Aluja, J. G. (1987). *Técnicas operativas de gestión para el tratamiento de la incertidumbre* [Operational management techniques for dealing with uncertainty]. Hispano Europea.

Keene, K. A. (2007). A characterization of dynamic reasoning: Reasoning with time as parameter. *The Journal of Mathematical Behavior*, *26*(3), 230-246. https://doi.org/10.1016/j.jmathb.2007.09.003

Larson, C., & Zandieh, M. (2013). Three interpretations of the matrix equation Ax= b. *For the Learning of Mathematics*, *33*(2), 11-17.

Lesh, R., & Kelly, A. (2000). Multitiered teaching experiments. In A. Kelly, & R. Lesh (Eds.), *Research design in mathematics and science education* (pp. 197-230). Lawrence Erlbaum Associates.

Mevarech, Z. R., & Kramarski, B. (2014). *Critical maths in innovative societies: The effects of metacognitive pedagogies on mathematical reasoning*. OECD. https://doi.org/10.1787/9789264223561-en

Mevarech, Z., Verschaffel, L., & De Corte, E. (2018). Metacognitive pedagogies in mathematics classrooms: From kindergarten to college and beyond. In D. H. Schunk, & J. A. Greene (Eds.), *Handbook of self-regulation of learning and performance* (pp. 109-123). Routledge. https://doi.org/10.4324/9781315697048-7

Oktaç, A. (2018). Conceptions about system of linear equations and solution. In S. Stewart, C. Andrews-Larson, A. Berman, M. Zandieh (Eds.), *Challenges and Strategies in Teaching Linear Algebra (*71-101*).* Springer. https://doi.org/10.1007/978-3-319-66811-6_4

Oktaç, A., & Trigueros, M. (2010). Cómo se aprenden los conceptos de álgebra lineal? [How to learn the concepts of linear algebra?]. *Revista Latinoamericana de Investigación en Matemática Educativa, 13*(4), 373-385.

Parra, V., & Otero, M. R. (2017). Enseñanza de la matemática por recorridos de estudio e investigación: Indicadores didáctico-matemáticos de las "dialécticas" [Teaching of mathematics by study and research paths: Didactic-mathematical indicators of "didactics"]. *Educación Matemática*, *29*(3), 9-49. https://doi.org/10.24844/EM2903.01

Robbins, S. P., & Judge, T. A. (2017). *Organizational behavior* (17th ed.). Pearson.

Rojas Garzon, P. J. (2015). Objetos matemáticos, representaciones semióticas y sentidos [Mathematical objects, semiotic representations, and meanings]. *Enseñanza de las Ciencias Revista de investigación y experiencias didácticas*, *33*(1), 151-165. https://doi.org/10.5565/rev/ensciencias.1479

Rondero, C., & Font, V. (2015). Articulación de la complejidad matemática de la media aritmética [Articulation of the mathematical complexity of the arithmetic media]. *Enseñanza de las Ciencias: Revista de Investigación y Experiencias Didácticas*, 29-49. https://doi.org/10.5565/rev/ensciencias.1386

Roth, W. M. (1996). Where is the context in contextual word problem? Mathematical practices and products in grade 8 students' answers to story problems. *Cognition and Instruction*, *14*(4), 487-527. https://doi.org/10.1207/s1532690xci1404_3

Serrano, L., Bosch, M., & Gascón, J. (2010). Cómo hacer una previsión de ventas: Propuesta de recorrido de estudio e investigación en un primer curso universitario de administración y dirección de empresas [How to make a sales forecast: Proposal for a study and research path in a first university course in business administration and management]. In A. Bronner, M. Larguier, M. Artaud, M., Bosch, Y. Chevallard, G. Cirade, & C. Ladage (Eds.), *Diffuser les mathématiques (et les autres savoirs) comme outils de connaissance et d'action* (pp.835-857). Montpellier.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification-the case of algebra. *Educational Studies in Mathematics*, *26*, 191-228. https://doi.org/10.1007/BF01273663

Smith, J., Lee, I., Zandieh, M., & Andrews-Larson, C. (2022a). Constructing linear systems with particular kinds of solution sets. In S. Karunakaran, & Higgins (Eds.), Proceedings *of the Annual Conference on Research in Undergraduate Mathematics Education*. National Science Foundation. https://doi.org/10.35763/aiem21.4237

Smith, J. L., Lee, I., Zandieh, M., & Larson, C. A. (2022b). A progression of student symbolizing: Solutions to systems of linear equations. *Avances de Investigación en Educación Matemática*, (21), 45-64.

Sokolowski, A., Yalvac, B., & Loving, C. (2011). Science modelling in pre-calculus: How to make mathematics problems contextually meaningful. *International Journal of Mathematical Education in Science and Technology*, *42*(3), 283-297. https://doi.org/10.1080/0020739X.2010.526255

Steffe, L. P., & Ulrich, C. (2020). Constructivist teaching experiment. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 134-141). Springer. https://doi.org/10.1007/978-3-030-15789-0_32

Steffe, L., & Thompson, P. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly, & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-306). Lawrence Erlbaum Associates.

Swan, M. (2020). Design research in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp.148-152). Springer. https://doi.org/10.1007/978-3-030-15789-0_180

Trabant, J. (1987). Louis Hjelmslev: Glossematics as general semiotics. In *Classics of semiotics* (pp. 89-108). Springer. https://doi.org/10.1007/978-1-4757-9700-8_4

Turgut, M. (2019). Sense-making regarding matrix representation of geometric transformations in $\mathbb{R}2$: A semiotic mediation perspective in a dynamic geometry environment. *The International Journal on Mathematics Education, 51*(7), 1199-1214. https://doi.org/10.1007/s11858-019-01032-0

Turgut, M., & Drijvers, P. (2021). Instrumentation schemes for solving systems of linear equations with dynamic geometry software. *The International Journal for Technology in Mathematics Education*, *28*(2), 65-80. https://doi.org/10.1564/tme_v28.2.01

Turgut, M., Smith, J. L., & Andrews-Larson, C. (2022). Symbolizing lines and planes as linear combinations in a dynamic geometry environment. *The Journal of Mathematical Behavior*, *66*(100948), Article 100948. https://doi.org/10.1016/j.jmathb.2022.100948

Van den Heuvel-Panhuizen, M. (2001). *Children learn mathematics*. Freudenthal Institute, University of Utrecht.

Van Den Heuvel-Panhuizen, M. (2005). The role of contexts in assessment problems in mathematics. *For the learning of mathematics*, *25*(2), 2-23.

Verschaffel, L., & De Corte, E. (2016). Word problems: A vehicle for promoting authentic mathematical understanding and problem solving in the primary school? In P. Bryant, & T. Nunes (Eds), *Learning and teaching mathematics* (pp. 69-97).

Verschaffel, L., Schukajlow, S., Star, J., & Van Dooren, W. (2020). Word problems in mathematics education: A survey. *The International Journal on Mathematics Education*, *52*(1), 1-16. https://doi.org/10.1007/s11858-020-01130-4

Wendell, R. E. (1997). Linear programming 3: The tolerance approach. In *Advances in Sensitivity Analysis and Parametic Programming* (pp. 137-157). Springer. https://doi.org/10.1007/978-1-4615-6103-3_5

Wittgenstein, L. (1953). *Philosophical investigations*. The MacMillan Company.

Yoneda, K., & Celaschi, W. (2013). A utility function to solve approximate linear equations for decision making. *Decision Making in Manufacturing and Services*, *7*(1-2), 5-18. https://doi.org/10.7494/dmms.2013.7.1.5

Zandieh, M., & Andrews-Larson, C. (2019). Symbolizing while solving linear systems. *The International Journal on Mathematics Education*, *51*(7), 1183-1197. https://doi.org/10.1007/s11858-019-01083-3