International Electronic Journal of Mathematics Education
2025, 20(1), em0801
e-ISSN: 1306-3030
https://www.iejme.com

MODESTUM

Research Article **OPEN ACCESS**

Problem creation to articulate proportional and algebraic reasoning

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Citation: Burgos, M., Tizón-Escamilla, N., & Chaverri, J. (2025). Problem creation to articulate proportional and algebraic reasoning. *International Electronic Journal of Mathematics Education, 20*(1), em0801. https://doi.org/10.29333/iejme/15650

ARTICLE INFO

Received: 09 Jul. 2024

Accepted: 31 Oct. 2024

ABSTRACT

This paper describes the design, implementation, and results of a training action with prospective primary education teachers, focusing on the creation of problems involving proportional and algebraic reasoning. Prospective teachers must solve a proportionality problem using both arithmetic and algebraic procedures, and then vary it to motivate proto-algebraic activity. Results show that participants successfully solved the task and created significant problems, which mostly motivated the expected algebraic activities. However, it can be observed that aspects of proportional reasoning or representations typical of proto-algebraic levels were not considered, which could have generated greater richness in the variety of problems created.

Keywords: proportionality, algebraic reasoning, teacher education, problem posing

INTRODUCTION

The relationship between the level of knowledge of the mathematics teacher and the learning achievement of their students justifies the interest in determining the type of didactic-mathematical knowledge that prospective teachers should possess. It is accepted that teachers must know the mathematics they are to teach their students. In particular, they must be competent to solve the problems that the curriculum proposes at the educational level at which they perform their teaching duties. However, there is also a broad consensus that these knowledge and competence do not guarantee effective teaching. The teacher requires specialized knowledge of the content itself, the transformations that must be applied to it in teaching and learning processes, and psychological and pedagogical factors, among others, that condition these processes. Therefore, from the research on mathematics education, it is necessary to design and implement formative experiences that characterize and promote the development of knowledge and professional competencies in teachers (Chapman, 2014; English, 2008; Ponte & Chapman, 2016).

Specifically, various researchers have highlighted the importance of incorporating problem creation into teacher training programs (Grundmeier, 2015; Malaspina et al., 2019). They recognize problem creation as an appropriate way to introduce prospective primary school teachers (PPTs) to mathematics teaching (Leavy & Hourigan, 2020), allowing them to deeply explore mathematical content, become aware of their potential shortcomings (Tichá & Hošpesová, 2013), and enhance their overall analysis of mathematical activity (Malaspina et al., 2019).

Considering problem creation as a means to assess, articulate, and develop the knowledge and competencies of mathematics teachers (Ellerton, 2013; Malaspina et al., 2015; Mallart et al., 2018; Milinković, 2015; Tichá & Hošpesová, 2013), this research describes the results of a formative intervention with PPTs, focused on creating problems involving proportional and algebraic reasoning.

Different theoretical perspectives and curricular proposals have been recommending the incorporation of algebraic content from the early educational levels with the aim of enriching school mathematical activity and facilitating access to mathematics in secondary education (Carraher & Schliemann, 2018; Kieran, 2022). Enhancing forms of algebraic reasoning in the early years of schooling requires a broader perspective on the nature of school algebra (Godino et al., 2014), understanding that algebraic reasoning occurs in "all activities aimed at developing in students an attitude to seek regularities, relationships, and properties, and to express them first in natural language and then in algebraic language" (Malara & Navarra, 2018, p. 54). According to Blanton et al. (2015), early algebra can be developed through five major ideas:

(1) generalized arithmetic,

(2) equivalence, expressions, equations, and inequalities,

(3) functional thinking,

(4) variable, and

(5) proportional reasoning.

Proportional reasoning emerges as a precursor to students' algebraic reasoning (Lundberg & Kilhamn, 2018). It involves "algebraically reasoning about two generalized quantities that are related in such a way that the relationship of one quantity to the other is invariant" (Blanton et al., 2015, p. 43). However, the development of proportional and algebraic reasoning in children requires specific initial training for prospective teachers.

Prospective teachers need to have knowledge of algebra and what its teaching entails in elementary school to later apply it in their practice and create teaching situations that develop their students' algebraic thinking (Branco & Ponte, 2012; Pincheira & Alsina, 2021). Research, such as that by Stylianou et al. (2019) and Ferreira et al. (2022) among others, highlights the positive effects of including algebraic practices in teacher training on students. In particular, proposing actions to promote generalization contributes to understanding the meaning of algebraic reasoning and how to foster it in elementary students (Ferreira et al., 2022).

Despite its importance, research results, such as those by Ferreira et al. (2022), Hohensee (2017), and Zapatera and Quevedo (2021) among others, show that prospective teachers' algebraic knowledge is insufficient. In the specific context of problem creation, Zapatera and Quevedo (2021) proposed that PPTs develop tasks to foster algebraic reasoning in elementary students based on two open-ended situations. However, almost all participants transformed both open-ended situations, which had multiple possibilities for promoting and developing algebraic thinking, into closed problems with a single solution, which they mostly solved arithmetic. The authors recommended including in teacher training programs experiences that allow prospective teachers to design tasks to detect and promote algebraic thinking in their students.

Similarly, research analyzing the didactic-mathematical knowledge necessary for teaching proportionality has shown the difficulties preservice or in-service teachers face in teaching concepts related to proportional reasoning (Buforn et al., 2020; Burgos & Godino, 2022; Hilton & Hilton, 2019; Weiland et al., 2020). The results of previous studies also revealed the challenges that prospective elementary and secondary school teachers may encounter when creating meaningful proportionality problems (Tichá & Hošpesová, 2013; Xie & Masingila, 2017). They propose problems that are not suited to the educational level, are incorrect, or are irrelevant to the development of proportional reasoning in their students (Burgos & Chaverri, 2022, 2023).

In this context, the results presented in this paper are part of a larger line of research aimed at designing, implementing, and evaluating formative interventions intended to develop proportional and algebraic reasoning in prospective elementary school teachers through the solving, analysis, and creation of tasks (Burgos & Godino, 2022; Tizón-Escamilla & Burgos, 2023).

THEORETICAL FRAMEWORK AND RESEARCH PROBLEM

In our research, we position ourselves from the perspective of the onto-semiotic approach (OSA) to mathematical knowledge and instruction (Godino, 2024). Below, we include the theoretical-methodological tools developed by this framework, which are essential to our research. We position problem creation as both an objective and a means for teacher training and define the research questions.

Pragmatic Meaning and Elementary Algebraic Reasoning Levels

From the pragmatist perspective of didactic-mathematical knowledge adopted by the OSA, the notions of mathematical practice and system of practices determine the starting point for analyzing mathematical activity (Godino, 2024). Mathematical practices, understood as any action aimed at solving a problem or performing a task, imply the capacity or competence of the person conducting the practice. However, competent execution of a practice involves the intervention of various interconnected objects that regulate and emerge from it. The term "object" is used broadly to refer to any entity that is in some way involved in mathematical practice and can be distinguished from others; the pragmatic *meaning* of an object is understood as the system of practices associated with the problem field from which the object emerges at a given moment.

Considering their nature and role in mathematical practices, the following typology of mathematical objects is proposed: problem situations, languages in their various registers, concepts, propositions, procedures, and arguments. The emergence of these primary objects occurs through the respective mathematical processes of problematization, communication, definition, formulation, procedural development (algorithmization), and argumentation (Godino, 2024). In particular, as a result of a generalization process, a type of mathematical object known in the OSA framework as an *intensive object* is obtained. This is the rule that generates the class (collection or set) of generalized objects and allows the identification of a particular element as a representative of the class. Through the particularization process, the new objects obtained are referred to as *extensive* (particular) *objects*.

Within the framework of OSA, a characterization of elementary algebraic reasoning (EAR) has been proposed, distinguishing four levels of reasoning (Aké et al., 2013; Godino et al., 2014), establishing criteria to identify purely arithmetic mathematical activity and to differentiate it from progressive levels of algebraization. The EAR levels model allows teachers to understand the characteristics of EAR through the identification of its characteristic objects and processes. The levels are defined based on the objects and processes emerging in the mathematical activity of a given individual when solving a specific problem. The criteria for distinguishing early EAR levels are, as follows:

- Level 0: Operates with first-degree generality intensive objects, using natural, numeric, iconic, or gestural languages.
- Level 1: Uses second-degree generality intensive objects, properties of the algebraic structure of $\mathbb{N}$, and equality as equivalence. Symbols may refer to recognized intensives, but without operating with these objects.

- Level 2: Uses symbolic-literal representations to refer to recognized intensive objects related to contextual information; solves equations of the form $Ax + B = C$ ($A, B, C \epsilon \mathbb{R}$)
- Level 3: Symbols are used analytically without reference to contextual information. Operations are performed using indeterminates or variables; solves equations of the form $Ax + B = Cx + D$ ($A, B, C, D \epsilon \mathbb{R}$).

Solving an equation of the type $Ax + B = C$ often involves simply reversing the operations; thus, it is considered an *arithmetic equation*. Contrarily, an equation of the form $Ax + B = Cx + D$ is termed an *algebraic equation* because solving it requires more than just undoing the indicated operation; it necessitates working with the unknown (Kilhamn et al., 2019).

The definition of algebraization levels is based on the presence of intensive algebraic objects (binary relations and their properties, operations and their properties, functions, and structures), the transformations applied to these objects (syntactic calculation, representation, generalization), and the type of language used. Level 0 indicates the absence of algebraic characteristics (arithmetic nature of the mathematical activity), while level 3 represents a consolidated algebraic activity (syntactically guided reasoning, actions on generalizations expressed in conventional symbol systems). Intermediate levels of proto-algebraic activity (level 1 and level 2) align with the proposal of early algebra (Carraher & Schliemann, 2018), distinguishing them from more established or consolidated forms of algebraic reasoning.

The application of EAR levels to practice systems related to proportionality tasks provides criteria to distinguish meaning categories in the progressive construction of proportional reasoning (Burgos & Godino, 2020), which is understood as the system of actions (operational and discursive practices) involved in solving proportionality problems. The arithmetic meaning of proportionality (EAR level 0) is characterized by the application of arithmetic calculation procedures (multiplication, division) to specific numerical values; no algebraic objects or processes are involved. The proto-algebraic meaning focuses on the concepts of ratio and proportion: recognizing the unit value in a unit rate procedure and using diagrammatic representations of solutions are associated with EAR level 1; solving a missing value problem through the formulation and solution of a proportional equation (where the unknown appears on only one side of the equation) is characteristic of EAR level 2. The algebraic-functional meaning, EAR level 3, is defined by the application of the concept of linear function and problem-solving techniques based on the properties of these functions.

Didactic-Mathematical Knowledge and Competencies of the Teacher

The model of categories of didactic-mathematical knowledge and competences of the teacher developed in the OSA (Godino et al., 2017) assumes that the teacher must have mathematical knowledge *per se*, coordinating common mathematical knowledge related to the educational level at which they teach, with an extended knowledge of mathematical content from higher levels. However, as any mathematical content is put into play, the teacher must have *didactic-mathematical knowledge* of the different facets that affect the planning and management of a specific mathematical topic: epistemic (about the content itself and the institutional reference meanings), ecological (guiding tasks according to the mandatory institutional curriculum), cognitive (understanding the student's thinking), affective (reacting to the anxiety, indifference, anger, etc., manifested by the students), interactional (identifying and responding to student conflicts and interactions), and mediational (choosing the most appropriate resources for teaching).

In this work, we focus primarily on the epistemic aspect of didactic-mathematical knowledge, which refers to the specialized knowledge about the content itself, that is, the particular way in which a mathematics teacher knows and understands mathematics. Teachers, in addition to the mathematics that allows them to solve problems, must be able to comprehend and mobilize the diversity of partial meanings of a specific mathematical object, solve mathematical tasks through different procedures, provide various justifications, and identify the knowledge involved during the process of solving a mathematical task.

The teacher must be competent to determine the configurations of mathematical objects and processes involved in the practices enacted in the intended meanings of the content (epistemic and institutional configurations), as well as the configurations that students use when solving problems (cognitive and personal configurations) (Godino et al., 2017). Moreover, teachers' recognition of the different levels of EAR when solving mathematical tasks and their ability to modify problems to promote algebraic reasoning in a specific content area (particularly proportionality) is considered a key aspect of the teacher's didactic-mathematical knowledge (Burgos & Godino, 2022; Godino et al., 2014).

Creation of Problems With Educational-Mathematical Purpose

Problem creation is a fundamental skill that enhances the didactic-mathematical knowledge of mathematics teachers; therefore, problem creation should be an objective in teacher training programs. Although different authors have given various names to the activity of creating problems, it essentially involves both the formulation of new situations and the reformulation of given problems. To determine what creating a problem entails, it is important to specify the elements that characterize a mathematical problem. According to Malaspina et al. (2015), these elements are, as follows:

(1) the *information*, that is, the quantitative or relational data provided in the problem,

(2) the *requirement*, what is asked to be found, examined, or concluded (this can be quantitative or qualitative, including graphs and proofs),

(3) the *context* (intra- or extra-mathematical) which determines the environment or scenario that gives rise to the mathematical activity, and

(4) the *mathematical environment* or mathematical structure that encompasses the mathematical concepts involved or that may be involved in solving the problem, their properties, and relationships.

Based on these elements, Burgos et al. (2024) proposed the following categories of problem creation:

- *Free or unstructured elaboration*. There is no initial (structured) problem situation, and no guidelines, instructions, or restrictions are provided regarding the components (context, environment, information, and requirements) of the new problem.
- *Semi-structured elaboration*. There is no base problem situation; however, information or restrictions about the elements of the new problem are included.
- *Structured elaboration or variation*. A problem is posed based on a previous one, such that the different elements (context, information, requirements, and mathematical environment) are known. Within the category of structured elaboration, it is considered to modify the information, change the requirements, exchange information and requirements, add new information, or pose new questions.

Teachers can develop problems, both in semi-structured and structured cases (variation), to address a didactic-mathematical purpose, such as involving certain mathematical objects (epistemic dimension) or responding to a certain level of cognitive demand (cognitive dimension), among others. Specifically, the competence of analyzing global meanings (identification and description of operative and discursive practices involved in mathematical activity) and the competence of onto-semiotic analysis of practices (recognizing the configuration of objects and processes involved and emerging from mathematical practices) of the teacher (Godino et al., 2017) are fundamental in creating problems for didactic purposes. Conversely, the creation of problems with a didactic purpose serves as a means to develop these competencies, since this process requires: reflecting on the overall structure of the problem (what it aims to achieve and whether the provided information is sufficient to solve it); analyzing possible ways to solve it and the mathematical objects and processes involved and how they relate (the mathematical structure); and recognizing possible difficulties that students may encounter and how to address them in the formulation of new situations.

Research Problem

The research problem addressed in this article is the design, implementation, and evaluation of a training activity for PPTs to integrate proportional and algebraic reasoning through the creation of problems. In this way, we continue the line of work initiated by Burgos and Godino (2022), and Tizón-Escamilla and Burgos (2023). Burgos and Godino (2022) required prospective teachers to create problems by structuring proportionality tasks in an arithmetic context so that the solution involved changes in the algebraic reasoning involved. In Tizón-Escamilla and Burgos (2023), prospective teachers were asked to create a problem that involved proportional reasoning from a semi-structured situation in a probabilistic context (run context) and then modify it to increase the algebraic reasoning involved. In both cases, participants had difficulties in creating meaningful problems that mobilized proportional reasoning to address a certain degree of algebraic reasoning, especially when this had to correspond to intermediate levels. These results led us to consider the need to

(1) reinforce the training of prospective teachers, both in problem creation and in the study of algebraic objects and processes (particularly concerning proportionality) and

(2) focus more specifically on the development of proto-algebraic mathematical activity, better adapted to primary education stages, where algebraic reasoning is incipient (Godino et al., 2014).

Specifically, we pose the following research questions:

1. Are PPTs capable of proposing different solutions to the same proportionality problem by exclusively involving arithmetic or algebraic procedures?
2. How do PPTs modify a proportionality problem so that its solution involves a different level of algebraic reasoning than the original problem?

METHODOLOGY

Given the nature of the research problem, we adopted the methodology of didactic engineering, understood in the generalized sense proposed by the OSA (Godino et al., 2013). This interpretation distinguishes four phases in the research: preliminary study (in its various epistemic-ecological, cognitive-affective, and instructional facets), experiment design (task selection, sequencing, and a priori analysis), implementation (observation and evaluation of the achieved learning outcomes), and retrospective analysis (derived from the contrast between what was planned in the design and what was observed during implementation). Additionally, content analysis (Cohen et al., 2018) is used to examine the response protocols of PPTs who participated in the training experience.

Research Context, Participants, and Data Collection

The training experience was implemented with 62 third-year students from the primary education degree program. During their degree studies, the student teachers have received specific training on epistemic aspects (mathematical knowledge), cognitive aspects (learning, errors, and difficulties), instructional aspects (tasks, activities, materials, and resources), and curricular aspects of mathematics teaching. At the time of developing the experience, it was expected that the participants would be able to put their acquired knowledge into practice to solve, design, and sequence mathematical tasks on specific content, in our case, proportional and algebraic reasoning.

Design and Implementation

The training activity was carried out over 6 sessions, each lasting two hours. **Table 1** describes the didactic trajectory (Godino, 2024, chapter 4) implemented and the content covered in the different sessions. These sessions are of two types:

Table 1. Implemented didactic trajectory

Session	Type	Contents
Session 1		
Analysis of mathematical practices	Theoretical-practical	• Mathematical practices • Mathematical objects and processes
Session 2		
Analysis of school mathematical tasks	Theoretical-practical	• School mathematical tasks • Characteristics of a task: mathematical environment, purpose, degree of complexity, potential resolution difficulties, material and temporal resources
Session 3		
Analysis of school mathematical tasks	Practical. Collaborative work	• Analysis of practices, objects, and processes in the solution of school mathematical tasks (some of them involving proportionality)
Session 4		
Creation of mathematical tasks	Theoretical-practical	• Elements of a task: context, environment, requirement, information • Ways to create mathematical tasks: free, semi-structured, structured • Didactic purpose in the creation of mathematical tasks
Session 5		
Algebraic sense	Theoretical-practical	• School algebraic reasoning • Algebraic objects and processes • EAR level model • Tasks to develop algebraic reasoning
Session 6		
Creation of mathematical tasks	Practical. Collaborative work	• Structured and semi-structured task creation in arithmetic, geometric, and probabilistic contexts • Didactic-mathematical epistemic purpose: (a) Use of the properties of the proportionality relationship (b) Develop algebraic reasoning
(Optional) Individual assessment task		

(1) theoretical-practical (large group) sessions where the theoretical content described in **Table 1** is presented and exemplified through guided work in small groups and collective discussion and

(2) practical sessions (in small groups of 4 or 5 students) where the participants work on the tasks provided by the instructor as a continuation of the previous theoretical-practical work.

Group work allows students to compare and enrich their proposals for various strategies to create and solve problems, identify potential difficulties within them, and assess their potential for developing algebraic thinking. After each practical session, students submit their work to the teacher (the first author of this paper); she reviews it and prepares a report on the task that she shares and discusses with the students in each group before the next class session. The feedback from the teacher consisted primarily of

(1) discussions with the students and evaluations of their work on the analysis of practices, objects, and processes in solving school mathematical tasks (reflecting on the typology and nature of mathematical objects and processes) and

(2) discussions about the significance of tasks created by the students (when a task is not significant and why) and their potential to meet educational objectives (involving specific mathematical content or addressing students' difficulties).

Instrument and Data Analysis

After these sessions, the PPTs worked individually on the task described below as an optional (voluntary) part of the final course assessment (**Table 1**). All students participated in the experience.

Consider the following problem:

> A student received a certain amount of money from their parents to cover meals for 40 days. However, they found a place where they could save 4 euros per day on food. As a result, the initial budget lasted 60 days. How much money did the student receive?
>
> a) Can the problem be solved by using purely arithmetic procedures? How?
>
> b) Can the problem be solved by using algebraic knowledge? How?
>
> c) Create variations of this problem, with each variation involving changes in the required level of algebraic reasoning. Solve these variations and justify the assigned level.

The goal of this task is to assess the flexibility of PPTs in solving problems using strategies of varying degrees of algebraic reasoning, as well as in modifying problems to induce changes in the algebraic nature of the practices they prompt. It also aims to evaluate their understanding of the curriculum and students' knowledge and how they apply this understanding in creating these problems.

Figure 1 describes the elements that characterize the base problem situation and illustrates the process of variation within it. Three variables are involved: the money received, the price of the daily menu, and the number of days the student can eat with

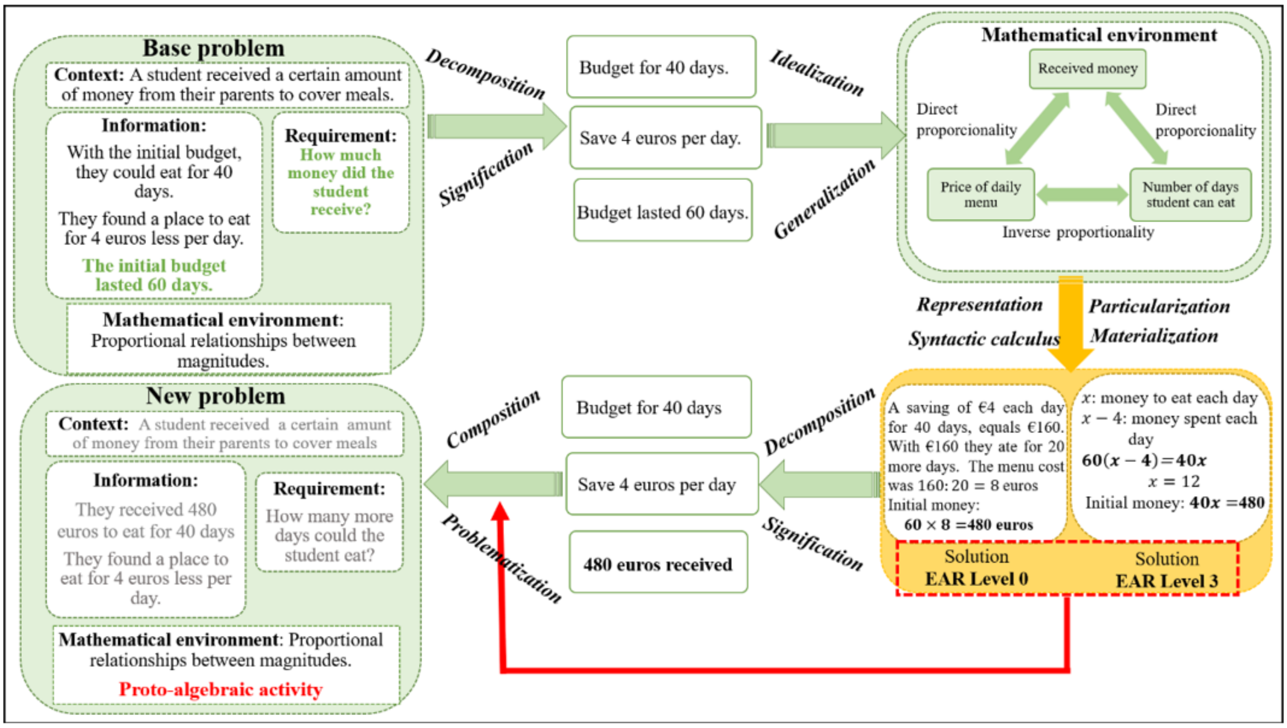

Figure 1. Diagram of the problem creation process through variation to address the didactic-mathematical purpose (Source: Authors' own elaboration)

the received money. Each pair of these variables is related either directly or inversely, while the other remains constant. Thus, when the money received is constant, the daily menu cost and the number of days the student can eat are inversely proportional. However, if the menu cost is constant, the number of days the student can eat is directly proportional to the money received for that purpose. The number of days covered by the initial budget, the difference between the initially planned menu cost and the price of the cheapest menu found (the savings), and the number of days the student could eat with this last menu are known. The amount of money received by the student is unknown. PPTs must interpret and decompose the problem text, giving meaning to the information and requirements. This allows them to place the involved quantities within the mathematical context (idealization, generalization) and understand how they relate to each other (mathematical structure). Solving the base problem (which involves processes of particularization, representation, and syntactic calculation, among others) provides new information. Analyzing the mathematical practices of the proposed solution (relationships between quantities, EAR level involved) enables the PPTs to decide how to modify the problem (which data to keep, what the new requirement and context will be) so that the new problem (composition, problematization) aligns with the established didactic-mathematical objective. Given that the initial problem needed to be solved through arithmetic procedures (EAR level 0) and algebraic procedures (EAR level 3), it was expected that PPTs would create problems through variation of the base, motivating an activity of a proto-algebraic nature (EAR level 1 and level 2).

The researchers conducted a descriptive analysis of a portion of the participants' reports, discussing possible discrepancies among themselves and collaboratively agreeing on the categories resulting from the analysis. After filtering the categories, all the PPTs' responses were reanalyzed.

For each problem created by the PPTs, we analyzed:

1. Whether it is significant. A problem is considered *significant* if the proposed statement genuinely presents a mathematical problem and the elements that characterize it are clearly identified. Specifically, the solution is not implied in the statement, it is possible to answer the requirement with the information given, its wording is clear, and it does not present any ambiguity. A problem is considered *partially significant* if it is clearly written, the solution is not implied in the statement, but it includes more information than necessary to solve the problem, which could hinder students' understanding. For example, the problem created by PPT24 is considered partially significant: "A student received €600 from their parents for food and €80 for transportation to the university canteen for 40 days. How much can they spend on food per day?" (the information about the money received for transportation is unnecessary to answer the question). In another case, a problem is considered *not significant* if its wording is confusing, it presents ambiguities, the solution is implied, or it lacks data necessary to solve it.
2. Whether it is a variation of the initial situation and of what type. A problem is considered a variation of the base problem if it shares any of its elements: context, information, requirement, and mathematical environment (**Figure 1**). Variations are classified based on which of these elements they modify and what type of problems they generate.
3. Whether its solution involves changes in the involved EAR level and how it does so.

Yes, the solution to the problem can be reached using arithmetic procedures, as follows:
First, we calculate the total savings the student makes daily. That is, 4 euros over the 40 planned days amounts to a total savings of 160 euros. With this amount, we can determine the number of extra days they can eat, in this case, 20 more days. The actual daily cost was then 160 euros over 20 days, which equals 8 euros per day. Since the total number of days was actually 60, the total budget would be 60 days multiplied by 8 euros per day, resulting in 480 euros.
In this case, we are directly working with specific natural numbers, to which we have applied arithmetic operations (specifically addition and multiplication).

Figure 2. Arithmetic solution to the base problem (PPT3) (Source: Authors' own elaboration)

Let z be the money received from the parents.
We denote by x the daily expense planned by the parents to cover meals for 40 days: $x = z / 40$.
And let y be the actual daily expense, which allowed for covering meals for 60 days: $y = z / 60$.
Thus, we have: $40x = 60y$.
We also know that $y = x - 4$, so if we substitute this in the previous equation, we get:
$40x = 60(x - 4)$, solving this we have: $40x = 60 \times 240$, then, $20x = 240$, and from here we find that $x = 12$.
Therefore, the amount received is $12 \times 40 = 480$ euros.

Solution b) corresponds to an EAR level 3 (we obtain an equation of the form: $Ax \pm B = Cx \pm D$).

Figure 3. Algebraic solution (EAR level 3) to the base problem (PPT61) (Source: Authors' own elaboration)

RESULTS

In this section, we present the analysis of the PPTs' responses to the proposed evaluation task. Although, due to space limitations, we do not show the results of the previous collaborative work sessions in this article, it is interesting to briefly mention the observations to provide a clearer idea of the progress of the participants. The evaluation of the reports produced by the PPTs in the sixth practice session (**Table 1**) revealed the difficulties PPTs faced in creating mathematical problems that involved proportional reasoning in different contexts. Less than half created relevant problems that involved proportional reasoning and met the didactic-mathematical objective: using properties of the proportionality relationship and distinguishing between proportional and additive situations. Furthermore, only one-fourth of the groups of prospective teachers created meaningful problems that activated algebraic reasoning and appropriately assigned their EAR levels.

Resolution: Arithmetic vs. Algebraic Strategies

Out of the 62 participants, one PPT did not solve the problem, and two others did so incorrectly (both in part a and b) because of a misinterpretation of the requirement, or the information provided.

First, the participants were required to solve the problem using exclusively arithmetic procedures. Of the 61 participants who responded to the prompt, one solved it correctly but used an algebraic method (the same as in the next item), and four explicitly stated that it was not possible to solve the task without resorting to algebraic practices. For example, participant PPT57 argued that "it is not possible since, without all the necessary data and the appearance of unknowns in the problem, it is impossible to calculate the amount of money received without using some level of algebraic reasoning." Except for one participant who used trial-and-error, the remaining 55 participants solved it in a manner similar to that of participant PPT3, whose solution is shown in **Figure 2**.

Although it was not required to justify the arithmetic nature of the proposed solutions, 15 PPTs were based, like PPT3 (**Figure 2**), on the intervention of particular natural numbers to which arithmetic operations (typical of an EAR level 0) were applied. Even though it is not common, some PPTs consider that the arithmetic solution is derived from the algebraic solution by translating the process followed in the algebraic solution into natural language and arithmetic operations.

Next, the PPTs had to propose solutions to the problem with algebraic features. Six PPTs followed proto-algebraic strategies: four of them adapted the same arithmetic strategy they had used in the previous section, using literal symbols to indicate unknowns and general relationships (for example D for the number of days, $D \times 4$ for the amount of euros saved, PPT20) but without operating with these (EAR level 1); the other two posed and solved the equation $\frac{x}{60} = \frac{4 \times 40}{20}$, where *x* is the amount of money received (EAR level 2). Of the remaining 55 PPTs who solved the problem, 24 proposed a system of equations (**Figure 3**), in which one equation establishes the relationship between the expected daily cost and the actual cost, and the other is an inverse proportionality equation that relates the price of the daily menu to the number of days it is possible to eat at that price.

The remaining 31 students directly formulated the equation $40x = 60(x - 4)$, indicating that *x* represents the initial amount of money planned for daily meals and that $x - 4$ is the actual amount spent on daily meals. In no case did they mention the inverse proportionality relationship between the magnitudes in their solutions.

Creation of Problems by Variation: Seeking Proto-Algebraic Activity

All participants except two created at least one problem. In fact, half of the PPTs proposed two, and three PPTs managed to develop three problems. As a result, a total of 96 problems were analyzed.

Table 2. Classification of problems by EAR level, significance, and variation (n = 96)

EAR	Significant		Partially significant	No significant	Total
	Variation	No variation			
Level 0	9	5	5	1	20 (20.83%)
Level 1	16	2	6	4	28 (29.17%)
Level 2	18	7	2	2	29 (30.21%)
Level 3	13	5	1	0	19 (19.79%)
Total	56 (58.83%)	19 (19.79%)	14 (14.58%)	7 (7.29%)	96 (100%)

The number of tables in a classroom is twice the number of chairs plus 6. If there are 36 pieces of furniture between tables and chairs in the classroom. How many tables and chairs are there?
Approach:
Tables: $2x + 6$ Chairs: x
We state the equation: "there are 36 in total"
$2x + 6 + x = 36$
We proceed to solve it:
$2x + 6 + x = 36$
$3x = 36 - 6$
$x = 30/3 = 10$
$x = 10$
Therefore, as a solution, we can say the following
Tables: $2x + 6 = 26$
Chairs: $x = 10$
The sum of tables and chairs is 36.
There are 10 chairs and 26 tables.
In this problem the assigned level would be level 3, since the unknown appears twice, so we have to "associate the part with x", $(2x + 6 + x = 36)$.

Figure 4. Problem proposed (no variation) by PPT30 (solution and assignment of EAR level) (Source: Authors' own elaboration)

Problem 1	Problem 2
A student received €480 to cover meals for 40 days. During those days they found a cheaper place where they could save some money on food. The budget lasted 20 days longer than planned. How much money did the student manage to save each day?	A student receives €480 in allowance from their parents to cover meals for 40 days. During those days they find a cheaper place where they save €4 per day on food. How many days could the student eat using the initial budget?

Figure 5. Problems proposed by PPT51 in categories 1.1 (problem 1) and 1.2 (problem 2) (Source: Authors' own elaboration)

To meet the objective of the task, the created problem had to be a variation of the base problem and motivate changes in the EAR level involved in the solution. **Table 2** summarizes the frequency of problems created according to their significance and whether they are variations or not, considering the EAR level involved in the solution of the new problem.

As shown in **Table 2**, most problems were significant. Those that were partially significant included unnecessary information, or the regularity condition that allowed solving the proportionality situation was not explicit. Non-significant problems could not be solved with the provided information.

Although 49 PPTs (81.67% of the participants) created at least one significant problem, 17 (27.42% of the total) did not succeed in doing so due to variations of the base problem. These PPTs developed problems that mostly led to the formulation of algebraic equations (which also did not prompt a change in the EAR level), as illustrated in **Figure 4**.

In contrast, 32 PPTs (53% of the total) created significant problems by varying the initial problem. The following categories have been identified in the significant problems created through variation:

1. *The form is maintained, but information and requirements are exchanged*. In this case, two types of exchanges occur:
   1.1. The received money becomes a known datum (information), and the question relates to the amount of daily savings (requirement) (problem 1 in **Figure 5**).
   1.2. The received money becomes a known datum (information), and the question arises about how many more days could be eaten (requirement) (problem 2 in **Figure 5**).
2. *The form is maintained; an element (information or requirement) is changed, while the rest remains the same*. Three situations are presented:
   2.1. Only the data are changed (**Figure 6**, problem a).
   2.2. The requirement is modified, changing the question to daily expenditure (sufficient to determine the received money) (**Figure 6**, problem b).
   2.3. A new requirement is added to the base problem (**Figure 6**, problem c).

Problem a (PPT50)	Problem b (PPT25)	Problem c (PPT19)
A student received a certain amount of money from their parents for their daily living for 50 days. However, some days they saved 5 euros. As a result, the initial budget lasted for 70 days. How much money did the student receive?	A student received a certain amount of money to cover meals for 40 days. However, they found places where they could save 4 euros per day on food. As a result, the initial budget lasted for 60 days. What is their daily expenditure on meals? Justify your answer.	A student received a certain amount of money from their parents to cover meals for 40 days. However, they found places where they could save 4 euros per day on food. As a result, the initial budget lasted for 60 days. How much money did the student receive? If instead of saving 4 euros, they saved another amount, how many more days could they stay?

Figure 6. Problems proposed in categories 2.1 (problem a), 2.2 (problem b), and 2.3 (problem c) (Source: Authors' own elaboration)

Problem a (PPT22)	Problem b (PPT49)	Problem c (PPT10)
A student received a certain amount of money from their parents to cover meals for 40 days. But, finally they found places where they could save, so, they could eat 20 days more. If we know that every day they spent 8 euros, how much money did the student receive?	A student received €480 from their parents to cover meals for 40 days. How much money will the student receive to cover meals for 60 days?	A student received a certain amount of money from his parents to cover meals for 40 days. If they receive 4 euros to cover meal one day, how much will the student spend in 40 days? How much money will the student need if they want to eat 100 days?

Figure 7. Problems proposed in categories 3.1 (problem a), 3.2 (problem b), and 3.3 (problem c) (Source: Authors' own elaboration)

PPT54. Problem 3 A student receives €480 from their family. They have been told that they can spend a maximum of €12 per day and a minimum of €8 per day. How long does their budget last if the student spends the maximum daily amount? And the minimum daily amount?

Figure 8. Problem proposed by PPT54, which belongs to category 4 (Source: Authors' own elaboration)

3. *The form of the base problem is changed by modifying both the information and the requirement* to transform it into a proportionality problem (missing value) between two quantities. The following problems are identified:
   3.1. The daily expenditure for a given number of days is known (information), and the budget is queried (**Figure 7**, problem a).
   3.2. The initial budget and the number of days it was intended for are known, and the money required for a different number of days is queried (**Figure 7**, problem b).
   3.3. The daily expenditure and the number of days for which the budget was intended are known, and the budget itself is queried (**Figure 7**, problem c).
4. *Distant variation*. Although the proportionality context is maintained, only the context and information are partially preserved, with the requirement being modified (**Figure 8**).

We indicate in **Table 3** the frequency of problems in each category.

Table 3. Frequency of types of significant problems created through variation according to their solution EAR level (n = 56)

Categories	Solution EAR level				Total
	L0	L1	L2	L3	
Maintain essentially the form of the base problem					
1.1. Exchange information for requirement (Data: Received money & Question: Daily savings amount).	0	3	2	1	6
1.2. Exchange information for requirement (Data: Received money & Question: How many days one could eat).	1	1	7	1	10
2.1. Change only data.	2	0	1	3	6
2.2. Change only requirement (daily expense, sufficient to respond to money received).	2	3	1	0	6
2.3. Add another requirement.	0	2	0	0	2
Modify the form of the base problem					
3.1. Information: Daily expenditure for a given number of days. Requirement: budget.	3	3	0	0	6
3.2. Information: Initial budget and number of days it was intended for (Requirement: Money needed for a different number of days).	1	6	2	0	9
3.3. Information: Daily expenditure, number of days for which the budget was intended (Requirement: Received money).	1	1	2	2	6
4. Distant variation. Partially retain context and information, change requirement.	1	2	2	0	5
Total	11	21	17	7	56

c) Algebraic level 2
A student received from their parents €480 to cover meals for 40 days. However, they found places where they could save €4 per day on food. As a result, the initial budget lasted more days. How many more days could the student eat?
Solution.
Initial budget €480
Days the student could cover meals with the initial budget 40
Save €4 per day
Total number of days they could cover meals → ?→"x"
Money to cover meals per day →? $= \frac{480}{40} = 12$
Money spent everyday $12 - 4 = 8$ euros.
$480 = 8x$; $\frac{480}{8} = x \rightarrow x = 60$ days could cover meals.
The level of algebraic reasoning is 2 because we use an equation $Ax + B = C$.

Figure 9. Problem 2 created by PPT13 (category 1.2) (solution and assignment of EAR level 2) (Source: Authors' own elaboration)

In section a), a level 0 of algebraic reasoning is carried out, as arithmetic operations with specific numbers are performed. In section b), a level 3 of algebraic reasoning is carried out, as the equations are symbolically formulated and a substitution technique is applied to solve them.
Therefore, we will approach other levels of reasoning by varying the proposed problem:

A student received €420 from their parents to cover meals for 45 days. If they spent €7 per day for the first 10 days, how much money do the student have to spend per day for the remaining 35 days?

To solve this problem, we will use a first-degree equation where the unknown x is the money that should be spent over the remaining 35 days. For this, in the equation, we must multiply the 10 days by the €7 spent per day and add to this the rest of the days, which are $35x$ (x is the money available per day to be spent), equating it all to the €420 that the student has. The equation would be as follows:
$10 \times 7 + 35x = 420$; $70 + 35x = 420$; $35x = 420 - 70$; $35x = 350$; $x = 10$.
Therefore, the student has 10€ per day to spend for the remaining 35 days.
The level of algebraic reasoning assigned to this problem is level 2, as operations with specific numbers are carried out, applying properties of the algebraic structure of N and equality as equivalence, as well as the use of a first-degree equation of the form $Ax + B = C$.

A student received €480 from their parents to cover meals for 40 days. If they need money for another 15 days, how much money will the student receive?

One way to solve this would be through a rule of three, where the unknown x is the money they will receive. If the student needs €480 for 40 days of food, they will receive x for 15 days, resulting as follows:
DAYS MONEY
40-------------- 480
15--------------- x
$x = (15 \times 480)/40 = 180$ euros will be received for those additional 15 days.
In this problem, level 1 of algebraic reasoning is employed, as a proportional relationship between the number of days and the total money is established, without operating with the unknown.

Figure 10. Problems created by variation by PPT11 (category 3.2) (solution and assignment of EAR levels) (Source: Authors' own elaboration)

As shown in **Table 3**, more than half of the significant problems created through variation essentially retained the form of the base problem. In these cases, the most common approach was to exchange information and requirement, so that the received money was known and the number of additional days for which one could eat was unknown. **Figure 9** shows the problem created by PPT13 in this category.

Additionally, as shown in **Table 3**, in 67.87% of the cases, the problems were designed to promote proto-algebraic mathematical activity (EAR level 1 and level 2), as requested. To "decrease the EAR level", PPT13 sought to ensure that the equation involved in the solution was arithmetic rather than algebraic, thus requiring an EAR level of 2 (**Figure 9**). In other cases, PPTs added information by considering that "by including the value of one of the unknowns that had to be calculated initially and thereby changing the information, we reduce and change the level of algebraic reasoning required" (PPT41).

When PPTs modified the form of the base problem (categories 3.1, 3.2, 3.3, and 4), they usually changed both the information and the requirement to transform it into a problem that establishes a direct proportionality relationship between two quantities: the received money and the number of days one can eat, with the daily menu cost being a fixed amount.

The situation exemplified in **Figure 10** is quite common: PPTs resorted to solving problems using the rule of three or cross-multiplication to find problems at EAR level 1. They identified the proportionality relationship and noted that the unknown is not operated upon, so the assigned level was appropriate for the proposed practice. However, in no case did they propose problems involving the comparison of ratios, which would imply practices characteristic of EAR level 1. Problems requiring determination of unit value (reduction to the unit), also characteristic of EAR level 1, were rare.

Regarding the identification of algebraic objects and processes, it was observed that, except for 5 PPTs who did not assign or justify the EAR level of the created problems, the other PPTs identified the EAR levels of the solutions proposed for their problems and did so correctly in most cases (80% of level 0 problems, 64.28% of level 1 problems, 82.76% of level 2 problems, and 68.42% of level 3 problems). Specifically, 59.37% of the created problems involved a change in the EAR level. It is also worth noting that the PPTs who created problems at EAR level 0 and justified it correctly had interpreted the prompt as requiring a problem that allowed for a non-algebraic solution rather than a solution of a different level from the two previously proposed solutions.

When PPTs did not correctly identify the EAR level in the solutions to their problems, it was usually because they considered that "equality as equivalence" is used when its meaning involved operations (thus recognizing it as level 1, which pertained to arithmetic practices) or they believed that the equation formulated was arithmetic when it was actually algebraic (assigning it level 2 when it corresponded to level 3). In particular, they focused on the final equation $Rx = S$, which resulted from transforming the equation $Ax + B = Cx + D$ or the system of equations, considering that no operations with the unknown were required; therefore, they assigned it an EAR level 2.

CONCLUSIONS

In addition to being competent in solving the problems they propose to their students, teachers must be capable of selecting, modifying, or creating them for specific educational purposes. Achieving this competence requires that teacher training programs include the design and implementation of specific actions (Malaspina et al., 2015). However, since creating problems leads prospective teachers to "rethink" the nature of mathematical objects before explicit instruction (Kılıç, 2017), these actions also contribute to the development of didactic-mathematical knowledge during initial teacher training.

In this study, we have presented the results obtained by a group of PPTs when varying a proportionality problem with the intention of promoting proto-algebraic activity (Godino et al., 2014). PPTs were required to first solve the base problem, proposing both an arithmetic and an algebraic solution to a proportionality problem, and then create new problems by variation whose resolution mobilized an algebraic activity of a level different from the previous ones.

Although we did not present the results of the previous collaborative work sessions in this article, we believe that the training received and the feedback provided by the teacher have helped the PPTs develop their ability to analyze mathematical activity, recognize the algebraic objects involved, and modify problems to address a specific EAR level. In fact, during the practical work conducted collaboratively prior to the assessment task, it was observed that PPTs had difficulties identifying mathematical objects in the practices, recognizing properties and relationships of proportionality, and creating meaningful significant problems that would engage algebraic reasoning. This was, in part, due to a poor and limited understanding of the transition from arithmetic to algebra. However, in the assessment task, the PPTs were able to create significant proportionality problems, recognizing that they promoted proto-algebraic mathematical activity. Specifically, regarding the first research question, the PPTs were able to provide two solutions to the original problem, successfully distinguishing between arithmetic and algebraic activities. These results significantly improve upon those obtained by Burgos and Godino (2022), and Tizón-Escamilla and Burgos (2023), who found greater difficulties in analyzing the algebraic activity involved in solving proportionality problems by prospective teachers. Second, most of the posed problems were meaningful, contrary to previous research findings that identified difficulties in creating meaningful proportionality problems by prospective teachers (Burgos & Chaverri, 2022, 2023). Moreover, a good part of the problems created were variations of the base problem that mobilized in its resolution an algebraic activity of a different level than the arithmetic or algebraic one (didactic-mathematical purpose of the proposed task). Additionally, the PPTs successfully identified the EAR level involved in the mathematical activity anticipated as a solution to their problems, which was justified based on the algebraic nature of the objects and processes involved. Again, these results represent a notable improvement compared to those obtained by Burgos and Godino (2022), and Tizón-Escamilla and Burgos (2023). In these previous experiences, in their attempt to achieve a specific EAR level, the PPTs sacrificed the meaningfulness of the statement (the solution was implicit in the statement, the problem made no sense, or the information provided by the statement did not allow answering the question). Moreover, they considered only "two degrees of algebraization: arithmetic (0) or algebraic (1), depending on the absence or presence of unknowns, regardless of how they were treated" (Burgos & Godino, 2022, p. 382).

Regarding the second research question, we observed that PPTs usually maintain the mathematical structure of the initial problem by exclusively modifying the information or requirement or exchanging both elements. To introduce a change in the EAR level, they aim for the solving equation to be arithmetic rather than algebraic (Kilhamn et al., 2019), in which case the EAR level is 2. When aiming for an EAR level of 1, although they pose missing value problems, they solve them using a rule of three without setting up the equation or operating with the unknown. Although the assigned level is adequate, from a didactic point of view, this practice should be avoided because it "hides" the intervention of ratios and proportion, leading to a "degenerate" meaning of arithmetic proportionality. In no case is the problem modified to seek a comparison of ratios, an essential component of proportional reasoning, nor are problems posed that are solved through the use of tabular (proto-functional in nature), diagrammatic, or graphical registers (Burgos & Godino, 2020), all of which have proto-algebraic characteristics.

Despite these results showing progress in the knowledge and competencies of the PPTs, we still found limitations that need to be addressed in future training designs. In varying the problems to motivate proto-algebraic activity, the focus was on the role of syntactic calculation involved (equations), neglecting other objects (relationships, functions) and processes (generalization, representation) that modulate the algebraic nature of mathematical practices, as well as the meanings inherent to proto-algebraic proportional reasoning. For example, sequences of proportional numbers and their representation using tabular registers establish a bridge between the proto-algebraic meaning centered on the idea of ratio and proportion and the strictly algebraic focus on the linear function (Burgos & Godino, 2020). In future implementations, it is necessary to:

- Reinforce the role of the comparison of ratios as an integral part of proportional reasoning (Buforn et al., 2020; Monje Parrilla & Gómez Alfonso, 2019) and highlight the connection between proportional and algebraic reasoning (Blanton et al., 2015).
- Allow PPTs to gain greater familiarity in their problems or solutions with informal and pre-symbolic representations (Hohensee, 2017), which are essential in the transition from arithmetic to algebraic thinking.

The formal algebra knowledge that prospective teachers may possess does not constitute sufficient preparation for teaching early algebra (Hohensee, 2017). Their experiences with algebraic practices are derived from considering equation solutions, which limits a broader perspective of algebra. The context of activities that can be conducted with representations such as informal diagrams and number lines can be a productive way to approach the pre-symbolic study of equations or functions (Hohensee, 2017). As part of flexible problem-solving and the ability to handle tasks involving different EAR levels that is, various degrees of formalization of mathematical activity, it is necessary for prospective teachers to understand the role of proto-algebraic representations, use them in their practices, and be able to design tasks that motivate their use.

Both the design of the training experience and the results obtained from its implementation with PPTs are relevant due to their applicability in teacher training programs in other contexts (different educational stages, different mathematical contents). The type of action described incorporates the resolution and creation of tasks in a way that is articulated with the analysis of mathematical practices, as a means to develop different types of reasoning, recognizing the link between the elements of the problem and the mathematical activity that it motivates (Malaspina et al., 2019; Mallart et al., 2018).

Author contributions: MB: conceptualization, formal analysis, writing — original draft preparation; **NT-E:** formal analysis, writing — original draft preparation; **JC:** writing — original draft preparation. All authors agreed with the results and conclusions.

Funding: This study is a part of the project PID2022-139748NB-100 funded by MICIU/AEI/10.13039/501100011033/ and by FEDER, UE, with the support of Research Group FQM-126 (Juanta de Andalucía, Spain).

Ethical statement: The authors stated that the study does not require any specific approval by any ethics committee, as no human and/or animal data are included in the study.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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