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High school Colombian students' variational thinking triggered by mathematical connections in a laboratory on linear functions

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ABSTRACT

The variational thinking of high school students based on mathematical connections was analyzed through a laboratory on linear functions. This qualitative research based on design was developed in three phases: diagnostic test, implementation of the mathematics laboratory and final test, with students from a public institution in Barranquilla, Colombia. The diagnostic test revealed difficulties in the concept of linear function and the modelling of situations. Therefore, a laboratory was implemented with eight activities focused on the concept of linear function, variation situations, generalization and conversion between different representations of the linear function (verbal, numerical, tabular, graphical, and algebraic). The results showed a significant advance in the variational thinking of the students, who managed to establish connections between different representations, procedural, metaphorical and better understand the meanings of the linear function. Also, the type of mathematical tasks used promote mathematical connections and variational thinking about the linear function and applications.

Keywords: variational thinking, mathematical connections, high school students and linear function

INTRODUCTION

Mathematical thinking represents an important cognitive activity in the teaching and learning processes of mathematics (Al Umairi, 2024; Eshaq, 2024; Stacey, 2005). According to Ministerio de Educación Nacional [Ministry of National Education] (MEN, 1998), it is classified into five types of thinking: metric, spatial, numerical, random and variational. The latter arises from the need to delve deeper into concepts, procedures and methods associated with variation, through quantities and magnitudes (Vasco, 2006). In addition, it involves cognitive processes that allow the analysis, organization and mathematical modeling of situations (Díaz, 2013; Vasco, 2006).

Additionally, variational thinking is related to observation, description, exploration, pattern identification, generalization, formulation of conjectures, representation, dependence and independence between variables and modeling (Nuñez-Gutierrez & Correa-Sandoval, 2019). From a mathematical perspective, this type of thinking is not isolated or developed independently, but its promotion involves the participation of other thoughts to ensure its development (MEN, 1998).

Based on the review of the specialized literature, there is research on variational thinking in primary school students (Alarcón et al., 2019; Doncel et al., 2021), secondary school (Mateus-Nieves & Moreno, 2021), high school (Ramos et al., 2015), higher level (Fiallo & Parada, 2014; Mendoza & Cabeza, 2017; Tejera, 2021), mathematics teachers in training (Amaya De Armas et al., 2016; Bonilla et al., 2015) and in-service teachers (Caballero & Cantoral, 2013). These investigations identified that the instruction of mathematical objects has been developed in a traditional way, that is, algebraic and algorithmic (Fiallo & Parada, 2014), possibly undervaluing the contributions of variational thinking. On the other hand, they have considered mathematical functions as useful fields for the study and promotion of variational thinking.

In this sense, difficulties associated with the analysis of the variation of phenomena in graphs of mathematical functions have been reported, which consists of the arbitrary assignment of a numerical value to the variables to perform arithmetic calculations, which is possibly linked to a lack of knowledge of the usefulness of the variable and its meaning in problematic situations (Ordóñez-Ortega et al., 2019). Also, the concept of function has been restricted to correspondence, without attending to variational language and thinking (Carabús, 2002). Other difficulties are related to the identification of what a function really is, its set-based approach, and the identification, construction, and manipulation of the multiple registers of representation of the function (Esquer-Armenta & Félix-Romero, 2021).

For his part, Díaz (2013) points out that, from the curricular point of view, the function is a fundamental thread because it runs from secondary school grades to university level, but students show difficulties linked to this mathematical content and are blamed for its complexity, generality, multiple representations and the variety of associated concepts. For this reason, Díaz (2013) delimits the errors and difficulties regarding the function in four groups:

(1) not knowing the domain and range of the functions,

(2) tendency towards regularity,

(3) a specific focus on the graphs, and

(4) a disconnection between the graphical and algebraic context.

Based on the above, the resolution of problem situations that involve the analysis of variation and change through different representation systems is essential (Fiallo & Parada, 2014).

Now, from a variational perspective, it is recognized that the study of functions is related to experimentation, reflection, construction of meanings and ways of expressing generality, as a result of mathematical modeling processes (Mateus-Nieves & Moreno, 2021). Likewise, aspects that are crucial for the understanding of the function must be considered, such as its interpretation represented by means of graphs, representations through formulas and tables, modeling of situations, mobilization between the multiple representations of the functions (Parra-Urrea & Pino-Fan, 2022), analysis of the effects of change in the parameters of the graphs of the functions and application of technology in the construction of the functions (Díaz, 2013).

In particular, studies report that concrete material and technological applications serve as mediators between the construction of knowledge in students, to enhance the consolidation of abstract mathematical concepts through the manipulation of objects, to which a meaning can be attributed (Ordóñez-Ortega et al., 2019). To do so, it is essential to promote enriched environments that allow the visualization, manipulation and construction of various conjectures about patterns and variation, such as mathematics laboratories, which have been used as a teaching strategy to improve learning in students (Bonham & Boylan, 2012). Moss et al. (2020) provide a learning progression to support early understanding of functions, where students analyze variables, linear relationships, and graphs of functions and implications to foster functional thinking. Hansen and Naalsund (2022) investigated opportunities and constraints related to in-service teachers' actions for the productivity of high school students' interaction patterns when solving a linear function problem.

It is worth mentioning that, in research on the understanding of linear equation and linear function (Barragán-Mosso et al., 2024; Hrnjičić & Alihodžić, 2024), difficulties are evident rooted in the failure to establish connections or translations between different representations and meanings (Esquer-Armenta & Félix-Romero, 2021), which motivates us to investigate this problematic direction. In fact, delving deeper into the field of mathematical connections, the literature on mathematics education recognizes studies focused on the derivative, concerned about the low understanding of this concept because students and some teachers have difficulties connecting meanings, multiple representations and solving application problems (García-García & Dolores-Flores, 2018, 2021; Rodríguez-Nieto et al., 2021a, 2021b, 2022a;, 2023a, 2023b, 2024).

Other research reveals his interest in the connection of reversibility between inverse functions such as exponential and logarithmic (Campo-Meneses & García-García, 2023; Campo-Meneseset al., 2021) the derivative and the integral (García-García & Dolores-Flores, 2021). Also, the connections between are important measurements of areas of flat figures (Caviedes-Barrera et al., 2019), connections between representations for the construction of algebraic language (De La Fuente & Deulofeu, 2022). The assessment of human or peoples' mathematics from the ethnomathematical connections in sociocultural environments (Rodríguez-Nieto, 2021; Rodríguez-Nieto & Alsina, 2022; Sudirman et al., 2024), using the trends in international mathematics and science study curriculum model to develop a coherent framework for building mathematical connections (Peters, 2024). For their part, Day et al. (2024) connected the most essential aspects of multiplicative thinking with algebraic, geometric and statistical reasoning and stated that, "exactly what these connections are and how they serve the goal of learning mathematics is rarely made explicit in curriculum documents with the result that mathematics tends to be presented as a set of discrete, disconnected topics" (p. 325).

Now, studies on connections are important, but there has been no evidence of research addressing students' work on variational thinking, connections, and linear function, but rather focusing on the knowledge of high school teachers based on connections (Bingölbali & Coşkun, 2016; Hatisaru, 2022). In this line, García-García (2024) investigated mathematical understanding based on mathematical connections made by Mexican high school students regarding linear equations and functions, where he refined the framework of mathematical understanding based on connections between different representations, procedural, among others. For example, one of the students has errors when working with the linear function because he has difficulties working with the slope of the line and the tabular record.

Specifically, regarding the two approaches of this research, it was identified that there is a current problem involving variational thinking and mathematical connections in the context of linear functions. On the one hand, it is recognized that secondary and university students have difficulties in understanding functions because they do not model when one variable changes as a function of another and in interpreting the different representation records (graphic, tabular, symbolic, and natural language), which refers to the failures to establish connections between multiple representations (Mateus-Nieves & Moreno, 2021). For example, Mateus-Nieves and Moreno (2021) state that, in solving a problem about situations of filling containers, the greatest difficulty for students was using the representation records (tabular and graphic) and interpreting, differentiating the notions of magnitude and variable, since they were confused, to the point of not establishing on which axis they should place the independent variable from the dependent one.

Furthermore, students have misconceptions about functions (López & Sosa, 2008), which causes difficulties in solving application problems where modeling connections emerge. For his part, García (2016) states that students' difficulties are compromised by the strategies and methodologies used by teachers where "data tabulation still prevails, without transcending to other representations and much less applying it in the modeling of phenomena that allow the student to adequately detail the variation and change in magnitudes of said phenomena" (p. 19). This, in turn, reflects the lack of modeling connections or other types of connections associated with the procedures that require changes in representation records.

Borke (2021) argues that prospective teachers have limited knowledge about the difficulties students face when learning the concept of function, which could hinder effective teaching of this and other concepts in calculus. Meanwhile, Tanışlı and Kalkan (2018) analyze how students interpret linear functions and slopes, highlighting that the type of reasoning used has a significant impact on the understanding of these topics. Kafetzopoulos and Psycharis (2022) suggest that visualizing the function as a covariational relationship between two quantities can be useful to overcome some of these obstacles. However, a continuing challenge is the diversity in the way students understand this relationship, which makes it difficult to achieve a more comprehensive and transferable understanding of the function concept. In this sense, the teaching of mathematical functions needs to adapt to the variability of students and consider how different forms of representation can be more effective depending on the context. A flexible and personalized approach could improve teaching and learning, allowing students to make more institutional and lasting mathematical connections with fundamental mathematical concepts.

On the other hand, from a curricular perspective, variational thinking requires the establishment of mathematical connections, because:

> […] combines the cognitive and the didactic to promote its genesis, enhancement and development. In this order of ideas, it is proposed that variational mathematical thinking should be considered as the basis on which the mathematical curriculum is structured, since it is a pillar and axis of other mathematical thoughts (numerical, spatial or geometric, stochastic, and metric) (MEN, 2014, p. 15).

Given the importance of variational thinking, mathematical connections, the relationships between these two approaches and the difficulties highlighted in the literature review, it is important to continue researching this topic in depth. Therefore, the objective of this research is to analyze the variational thinking of high school students based on mathematical connections through a laboratory on linear functions.

CONCEPTUAL FRAMEWORK

The research is based on four theoretical elements: variational thinking, representation, variational tasks (VTs) and mathematical connections.

Variational Thinking

For the purposes of this research, variational thinking it is a dynamic form of thinking that seeks to produce, from the mental, systems that establish relationships and/or connections between internal variables and patterns of covariation in subprocesses extracted from reality (Vasco, 2003). In this line, variational thinking requires the establishment of connections between metric and numerical reasoning, for example, when measurements are used. In addition, it requires the ability to think in spatial terms if one or more of the variables are of a spatial nature. Analytical and representational systems are also fundamental, although logical systems, set theory or other types of systems can be used to establish connections between representations, relations and general transformations of mathematical objects (Vasco, 2003).

Furthermore, Vasco (2003) points out that variational thinking as a cognitive process can occur in four moments:

(1) capturing what changes, what remains constant and the patterns that are repeated in certain processes,

(2) production of systems or mental models whose internal variables interact in such a way that they behave with some approximation of the identified variables,

(3) functioning of mental models to study what results they produce, and

(4) comparison of the results with what occurs in the modeling process, and if possible, reviewing and refining the model, or discarding it and starting the process again.

In this aspect, variational thinking is related to the recognition, perception, identification and characterization of variation and change in different contexts as well as in the description, modeling and representation in different systems or representations such as verbal, iconic, graphical or algebraic (MEN, 2006).

Other authors have worked on variational thinking and state that it is the field that allows the study of the phenomena of teaching, learning and communication of mathematical knowledge about variation and change (Dolores-Flores & Salgado, 2009). This thinking involves the development of strategies, forms of reasoning, elements and linguistic structures that allow communicating the study and analysis of variation and change, in addition, developing thought structures that promote the identification, analysis and interpretation in a natural way, of situations that imply change, modeling and transformation to others (Mendoza & Cabezas, 2017).

$$f(x) = x + x + 2;$$
$$f(x) = 2x + 2;$$
$$f(x) = x^2 - x^2 + 2x + 3 - 1$$

Figure 1. Processing between algebraic representations (Nuñez-Gutierrez, 2018)

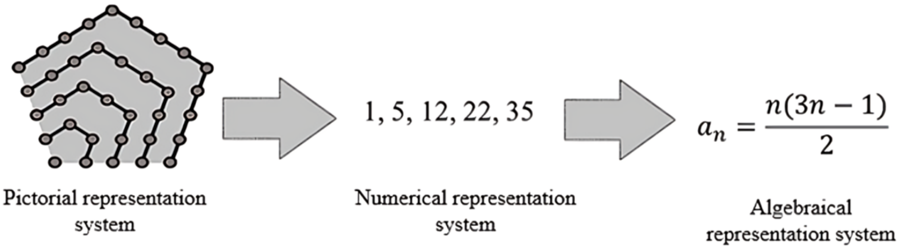


Figure 2. Conversion between representation systems for pentagonal numbers (Nuñez-Gutierrez, 2018)

Variational tasks

VTs are actions, activities, and executions that facilitate the organization of the study of variational thinking within a specific situation, with specific actions and objectives in particular contexts (graphic, analytical, numerical, among others) (Caballero & Cantoral, 2013).

Representations

Representations are mental or physical schemes that facilitate manipulation, operation and conversion, based on the variety of properties and structural relationships of mathematical concepts and ideas they contain (Lupiáñez, 2016). These actions are called processing in a system and conversion between systems. The first refers to the transformations of representations within the system in which they were created (Lupiáñez, 2016). For example, these three different expressions (see **Figure 1**) are equivalent algebraic representations that can be derived from each other; for this, mathematical operations are used to demonstrate processing and establish equivalences between them.

The second is the translation of an expression in a certain system into another expression in another different system (Lupiáñez, 2016). An example is the conversion between the representations of pentagonal numbers (from pictorial to numerical and from numerical to symbolic or algebraic). Nuñez-Gutierrez (2018) points out that these conversions involve mathematical procedures, one of which consists of counting the points to calculate the k-ésimo terms of the succession of pentagonal numbers. These forms of representation are equivalent to each other (**Figure 2**).

These types of representations are evident in the connection theory from the perspective of Businskas (2008), which are called different representations classified as equivalent when the representation register does not change or is maintained (graphic-graphic) and alternate when the representation register changes, for example, in **Figure 2** three representations are observed that refer to alternate connections (pictorial-numerical-algebraic or symbolic). That is, the representation systems from Lupiáñez (2016) are consistent with the types of representations used in extended theory of connection (ETC), which will be shown below. In terms of semiotic representations theory, alternate representations refer to conversions and representations equivalent to treatments (Duval, 2006).

Extended Theory of Connections

In this research a mathematical connection is understood as "a cognitive process by which a person relates two or more ideas, concepts, definitions, theorems, procedures, representations and meanings with each other, with other disciplines or with real life" (García-García & Dolores-Flores, 2018, p. 229). There are proposals of theoretical models for its characterization, among them, that of Businskas (2008), which has been used most frequently in studies oriented towards mathematical connections, which can be intra-mathematical connections that "are established between concepts, procedures, theorems, arguments and mathematical representations with each other" (Dolores-Flores & García-García, 2017, p. 160), and extra-mathematical connections where "a relationship is established between a mathematical concept or model and a problem in context (non-mathematical) or vice versa" (Dolores-Flores & García-García, 2017, p. 161). In this research, the following categories of mathematical connections are considered (**Table 1**).

It is worth noting that mathematical connections are evident in the written, verbal and gestural productions that people make when solving a certain mathematical task (García-García & Dolores-Flores, 2019). In addition, in Rodríguez-Nieto et al. (2022a) a theoretical articulation was carried out with the onto-semiotic approach (OSA) where the structure of the mathematical connection was detailed in terms of practices, processes, objects and semiotic functions that relate them with the objective of recognizing a type of mathematical understanding. Especially, the mathematical connection in terms of the semiotic function (Font, 2007) is structured from an antecedent (expression: initial object) related to a consequent (content: final object), which is supported by a correspondence code that, in other words, refers to the argument or justification that validates a procedure or representation that has been carried out.

It should be noted that, in this research, a definition of mathematical connections understood from the perspective of García-García and Dolores-Flores (2018) is used because the tools of the OSA will not be considered for the analysis of the data and the

Table 1. Types of mathematical connections

Mathematical connections	Description
Modelling	Relationships between mathematics and real life, which are evident when the subject solves non-mathematical or application problems where he has to propose a mathematical model or expression (Evitts, 2004).
Procedural	They originate when the subject uses rules, algorithms or formulas to complete or solve a mathematical task. This type of connection is formal, A is a procedure used to work with concept B (García-García & Dolores-Flores, 2019).
Different representations	When the subject represents a mathematical concept through alternate or equivalent representations (Businskas, 2008; García-García & Dolores-Flores, 2019). Equivalents are transformations of representations made within the same representation or register. Alternate representations refer to representations where the register in which they were constructed is modified.
Feature	They are identified when the subject shows characteristics of mathematical concepts or descriptions of their properties in terms of other concepts, which make it different or similar to other concepts (Eli et al., 2011; García-García, 2019; García-García & Dolores-Flores, 2019).
Reversibility	This type of connection occurs when a subject starts from concept A to reach concept B and reverses the process starting from B to return to A (García-García & Dolores-Flores, 2019).
Part-whole	They are evident when the subject establishes logical relationships in two ways (general-particular and inclusion). The generalization relationship is of the form A is a generalization of B, and B is a particular case of A (Businskas, 2008; García-García & Dolores-Flores, 2019).
Meaning	These are identified when a subject "attributes a meaning to a mathematical concept, that is, what it means to him […]. This includes those in which a subject gives a definition that he has constructed for these concepts" (García-García & Dolores-Flores, 2020, p. 5). This connection is identified when the meaning is used to solve a problem (García-García & Dolores-Flores, 2019).
Implication	These are identified when a concept P leads to another concept Q through a logical relationship ($P \rightarrow Q$) (Businskas, 2008). To identify this type of connections, the first linguistic form was proposed by Selinski et al. (2014), who emphasized that "a student makes a connection with an explicit logical implication. Logical implications were used to write if-then or to link words such as when, why, should, etc." (p. 559).
Metaphorical	They are understood as the projection of the properties, characteristics, etc., of a known domain to structure another less known domain (Rodríguez-Nieto et al., 2022b).
Metaphorical connections on mnemonics	This connection is "understood as the relationship established by the subject between a mnemonic rule (often a familiar resource) and a mathematical object, rule, or mathematical procedure to memorize and use strategically more easily" (Rodríguez-Nieto et al., 2024, p. 18). These types of connections are inclusive and recursive where three elements must be considered: (a) keywords that are similar to the word (or term) being referred to, (b) acronyms it is identified when the first letter of each word is used in a list to construct another word, and (c) acrostics which consist of constructing a sentence, where the first letter of each constitutes the term studied.

results. Therefore, only the types of connections activated in the resolution of the VTs by the students will be mentioned without considering the details of practices, processes, objects and semiotic functions (Rodríguez-Nieto et al., 2024).

Relationship Between Variational Thinking and Mathematical Connections

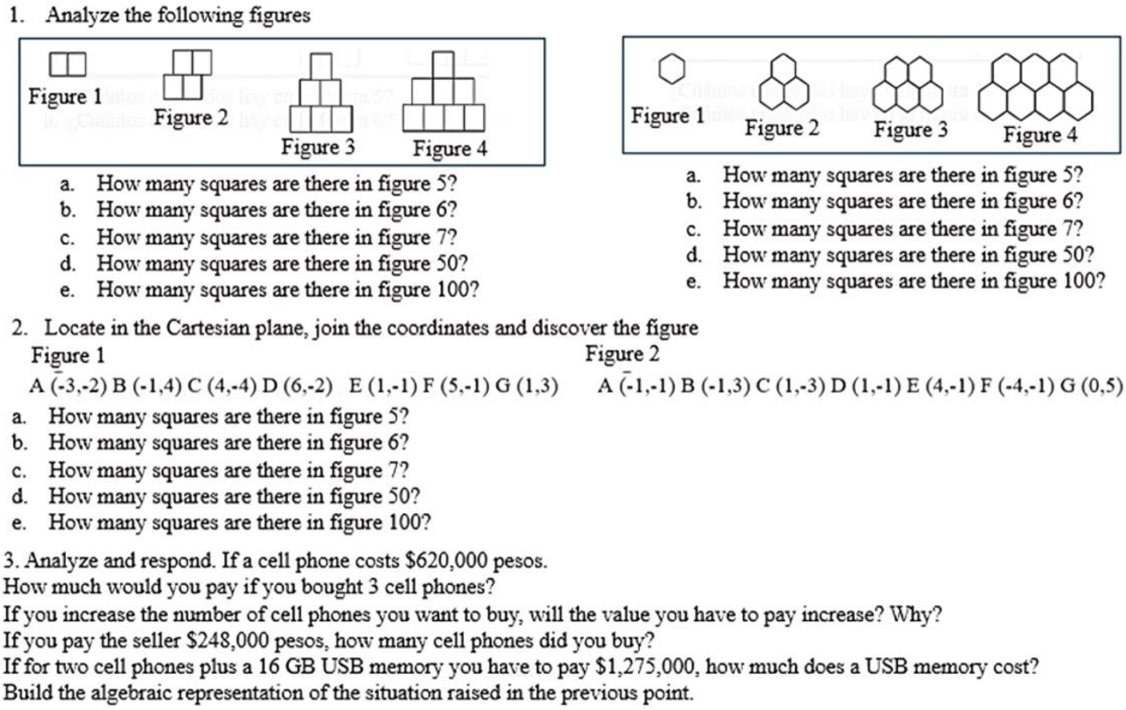
Initially, it can be stated that variational thinking seeks to activate and develop mathematical modelling processes. In fact, to solve a challenging problem, it is essential to build a model of the situation in which the variables interact in a similar way to those of the original context, which requires activating variational thinking. This thinking includes the modelling process, which can be divided into several phases: first, the identification of variation patterns (what changes and what remains constant); then, the creation of a mental model; then, the implementation of the model; followed by the comparison of the results with the modelled process, and finally, the revision of the model. These phases do not necessarily follow a strict sequential order and can be interconnected through feedback loops (Vasco, 2003).

In this whole process, the mathematical connection of modeling is fundamental (Evitts, 2004; García, 2024; Ledezma et al., 2024), since it allows establishing numerical relationships that describe the interactions and changes between the variables. A correct mathematical connection ensures that the model is consistent with the reality it represents, and, in turn, connections must be established: procedural, feature, implication, among others. In addition, the connections facilitate the understanding (Berry & Nyman, 2003; Businskas, 2008) and validation of the model, since it allows comparing and adjusting the results obtained in the different phases of the process. In short, it is ensured that the mathematical connection is key to guarantee the precision and effectiveness of the variational approach in solving the problem.

In terms of ETC, processing in a system refers to connections of equivalent different representational types and conversion or translation between systems to connections of alternating different representations. Another important aspect of variational thinking is that it seeks to develop the teacher's competence in teaching mathematics in a dynamic, articulated and rich way in the assessment of prior knowledge, which is evident in ETC because the instruction-oriented connection considers the activation of prior knowledge to address new knowledge or concepts (Businskas, 2008; Cantillo-Rudas et al., 2024).

METHODOLOGY

It is a design-based research (Godino et al., 2013; Wittmann 1995) carried out through the development and implementation of a mathematics laboratory, characterized by a series of VTs that promoted the development of variational thinking in high school students through the linear function.



1. Analyze the following figures

Figure 1 Figure 2 Figure 3 Figure 4

a. How many squares are there in figure 5?
b. How many squares are there in figure 6?
c. How many squares are there in figure 7?
d. How many squares are there in figure 50?
e. How many squares are there in figure 100?

Figure 1 Figure 2 Figure 3 Figure 4

a. How many squares are there in figure 5?
b. How many squares are there in figure 6?
c. How many squares are there in figure 7?
d. How many squares are there in figure 50?
e. How many squares are there in figure 100?

2. Locate in the Cartesian plane, join the coordinates and discover the figure

Figure 1
A (-3,-2) B (-1,4) C (4,-4) D (6,-2) E (1,-1) F (5,-1) G (1,3)

Figure 2
A (-1,-1) B (-1,3) C (1,-3) D (1,-1) E (4,-1) F (-4,-1) G (0,5)

a. How many squares are there in figure 5?
b. How many squares are there in figure 6?
c. How many squares are there in figure 7?
d. How many squares are there in figure 50?
e. How many squares are there in figure 100?

3. Analyze and respond. If a cell phone costs $620,000 pesos.
How much would you pay if you bought 3 cell phones?
If you increase the number of cell phones you want to buy, will the value you have to pay increase? Why?
If you pay the seller $248,000 pesos, how many cell phones did you buy?
If for two cell phones plus a 16 GB USB memory you have to pay $1,275,000, how much does a USB memory cost?
Build the algebraic representation of the situation raised in the previous point.

Figure 3. Diagnostic test applied to students (Source: Authors' own elaboration)

Participants and Context

The study included four researchers who acted as coordinators and instructors of the mathematics laboratory and thirty-seven ninth grade students from a public institution in Barranquilla, Colombia. The students' ages ranged between 14 and 18 years. According to the Colombian school curriculum, the topic of linear function is taught in ninth grade, so this population was selected for the research. In addition, during the period of the study, the students had not been taught about linear functions. The research was developed in three stages:

(1) application of a diagnostic test,

(2) design and implementation of the mathematics laboratory, and

(3) application for a final test.

Diagnostic Test

It was carried out with the objective of examining the initial situation of the students in relation to their knowledge and abilities on the linear function and to compare the changes, if any, at the end of the implementation of the mathematics laboratory. The diagnostic test was based on questions about the generalization of linear patterns that correspond to the form $2x$ and $2x - 1$, location and graphical representation on the Cartesian plane and resolution of problem situations through modelling (see **Figure 3**). Also, this test was designed to somehow show the connections that students establish in solving tasks, for example, to investigate what connections (procedural, representations, or others) students activate to favor their development of variational thinking, in this case, what changes, what remains constant and the patterns that are repeated in certain processes are considered, for example, to obtain the figures requested in the tasks. In addition, reaching a generalization by going through different representations and procedures.

The diagnostic test was validated by experts in the field of pattern generalization and mathematical connections (three researchers, two Colombians and one Mexican), such that, it was agreed that these tasks promote connections (Rodríguez-Nieto et al., 2024) and moments of variational thinking (Vasco, 2003).

Mathematics Laboratory

The mathematics laboratory is a pedagogical strategy that contains a set of mathematical activities (VTs) that involve the use of different materials, which must be developed autonomously by the participants, in order to contribute to the construction and foundation of mathematical thinking (Pabón & Gómez, 2008).

This strategy was developed in three blocks (see **Table 2**) with one session per activity in which the following were implemented:

(I) three activities on the concept of function,

(II) two activities on the concept of linear function, and

(III) three activities on the modeling of variation and change situations through different representations: verbal, tabular, numerical, algebraic, and graphical of the linear function.

Table 2. Activities developed in the mathematics laboratory

Blocks	Activities
I	VT 1. Human Cartesian plane
	VT 2. Relate and function
	VT 3. Coordinates on the Cartesian plane
II	VT 4. Crossword puzzle with terms used in linear function
	VT 5. Find your functional partner
III	VT 6. Variation situations
	VT 7. Application and generalization of the linear function
	VT 8. From graphical representation to algebraic representation

Table 3. Description of activities

Block	Session	VT	Objective	Materials	Actions
I	1, 2, & 3	VT1. Human Cartesian plane VT2. Relate and function VT 3. Coordinates on the Cartesian plane	Identify the concept of function through graphical representation both in the Cartesian plane and in the sagittal diagram	• Cardboard • A meter stick • Ruler • Board • Styrofoam • Colored markers • Adhesive tape • Waterproof tape • Colored rubber bands • Pencil • Scissors • Pearl-headed pins	• Locate the indicated coordinates with their bodies • Identify whether the graphical representation is a function • Relate the starting set to the ending set • Locate the coordinates on the icopor board • Identify the representation on the Cartesian plane
II	4 & 5	VT 4. Crossword puzzle with terms used in linear function VT 5. Find your functional partner	Define the generalities related to the concept of function and linear function	• Cardstock • Modeling clay • Marker • Bond paper • Scissors • Adhesive tape • Blank sheets of paper • Crossword puzzle printed on tabloid paper • Block sheets of paper	• Match the term with its respective definition • Complete the crossword puzzle with terms used in linear function
III	6, 7, & 8	VT 6. Variation situations VT 7. Application and generalization of the linear function VT 8. From graphical representation to algebraic representation	Model situations of variation and change based on geometric, tabular, algebraic and graphical representation that facilitate the development of variational thinking	• Cardboard • Bond paper • Adhesive tape • Whiteboard • Erasable marker • Non-commercial Colombian bills • Pencil • Graph paper • Ruler • Marbles • Popsicles • Plastic cups • Rubber bands • Squared sheets • Puzzles • Scissors	• Complete the table of values by obtaining the numerical values of the situations presented • Create a graphical representation based on the problem situation • Obtain the pattern from a situation • Obtain the pattern from a graphical representation

Activity I and activity II focused on the conceptual mastery of the function and the linear function, to enhance variational language and III focused on promoting variational thinking through mathematical connections in the generalization of patterns and the analysis of variation in the different representations.

Each mathematics laboratory session lasted 90 minutes and consisted of three moments:

(1) the instructors gave each student an instruction manual that explained the actions they should perform during the session, and also recorded the objective, the materials to be used, and the task to be solved (**Table 3**),

(2) the students performed the tasks individually or in groups depending on the activity, and

(3) the students voluntarily participated at the blackboard to explain to the group how to solve the tasks.

Variational Tasks

VT 1. Human Cartesian plane

In this task, the classroom floor is used as a stage to simulate a Cartesian plane, which was made with cardboard with axes measuring 630 cm and distributed in 30 cm for each unit. Students must perform mathematical calculations to obtain the coordinates, locate them on the plane, build the graphical representations and identify if they represent a function.

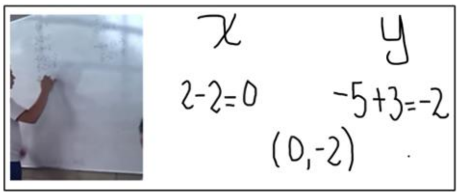


Figure 4. Mathematical calculations performed by students (Source: Field study)

Figure 5. Location of students on the Cartesian plane (Source: Field study)

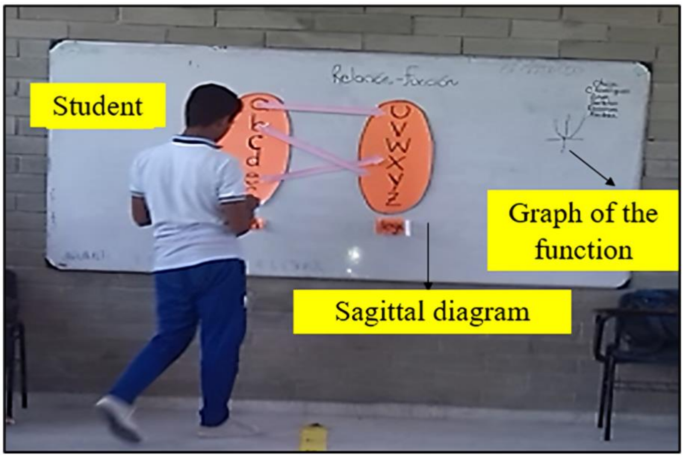


Figure 6. Representation in the sagittal diagram (Source: Field study)

For the mathematical calculations, two dice made with sheets of paper are used; the first dice has the integers 1 to 6 on each of its faces and the second, the numbers from -6 to -1. Students must roll both dice at the same time, and with the numbers obtained, perform the operation indicated on the board, where the first calculation is the location on the abscissa axis and the second, on the ordinate axis. Then, from the coordinate found, the student proceeds to place his body on the coordinate (see **Figure 4**).

In the graphical representation, students are organized into groups and randomly select a card that indicates the coordinate to be located on the Cartesian plane. After each student in the group is located on their coordinate, they must analyze the graphical representation that is being formed with their bodies and identify whether it corresponds to a function, through the vertical line test (**Figure 5**).

VT 2. Relate and function

A sagittal diagram was made on cardboard with two sets, the starting set and the ending set, whose elements are the letters of the alphabet. This diagram is placed on the classroom blackboard so that it can be viewed by all students, and they are encouraged to participate. Based on the diagram, the instructors provide an interactive explanation to the students about the concept of function and reinforce the property that every function is a relation but not every relation is a function (activating the feature type mathematical connection). Subsequently, the students are challenged to relate elements of the starting set (domain) with the ending set (range) by means of arrows, also made with cardboard, and to identify whether these representations correspond to a function or only to a relation (**Figure 6**). In addition, they establish mathematical connections of different types of alternate representations where they move in graphic records and the function diagram.

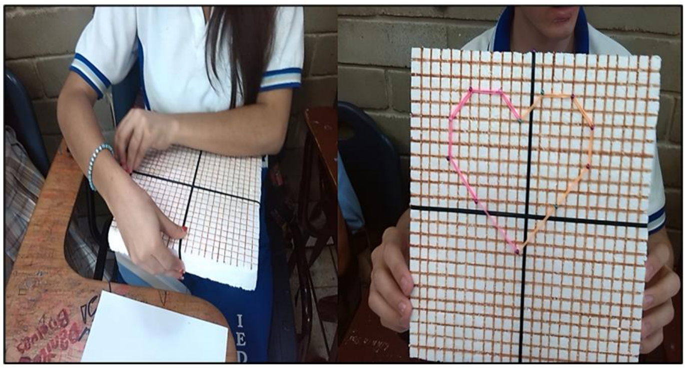


Figure 7. Representation on the Cartesian plane (Source: Field study)

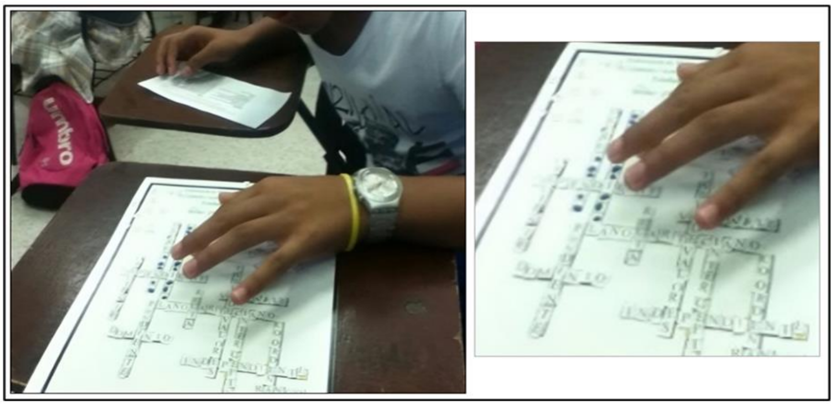


Figure 8. Crossword puzzle of concepts related to linear function (Source: Field study)

VT 3. Coordinates on the Cartesian plane

For this task, each student is given manipulative materials for the location of coordinates and graphic representation on the Cartesian plane. These are a styrofoam board or styrofoam type IV (30 cm * 30 cm), a meter of water tape, colored rubber bands, pins with colored pearl heads and a ruler. In addition, the instruction manual contains coordinates for the construction of a graphic representation. It should be noted that seven different representations were provided, each distributed in a different instruction manual, so that they can be exchanged between students during the activity, at the time of finishing with a representation.

The task is for the student to draw squares of equal size on the Styrofoam board, then use the water tape to define the *x-axis* and *y-axis* in a simulation of a geoboard, then locate the coordinates initially given in the instruction manual with the pins, and finally, use the colored rubber bands to join the coordinates to build the representation. The student must establish whether these constructions represent a function (see **Figure 7**).

VT 4. Crossword puzzle with terms used in linear function

The laboratory is a set of strategies that can be didactic, technological, playful, among others, which is why a crossword puzzle was implemented for this activity to reinforce the terms and definitions associated with the linear function. Before starting this task, the instructors make a presentation on the terms and definitions related to the linear function, with the purpose of having the students master the variational mathematical language through the persuasive power of the connections of meaning linked to the question: What does a mathematical object mean? Subsequently, the students are given the instructions that only contain the definitions of the following terms: linear function, slope, intercept, variable, dependent variable, independent variable, table of values, Cartesian plane, coordinates, straight line, domain, range, increasing function and decreasing function. In addition, they are given a crossword puzzle printed on tabloid paper with six horizontal lines and eight vertical lines of boxes. For the location of each term, they are given a bag with all the letters that make up the terms used, which must be placed on the tabloid paper correctly with plasticine, to form the word of the term and thus complete the crossword puzzle (**Figure 8**).

VT 5. Find your functional partner

In this task, the group participation of all the members of the mathematics laboratory is encouraged. To do this, cards printed on cardboard with the definitions and terms mentioned above are placed on the board with adhesive tape. The definitions are printed on yellow cardboard and the terms are printed on green. These cards are placed on the back so that the students can discover them. The activity consists of picking up a green card (term) and then a yellow one (definition) or vice versa to obtain a pair. After having a pair of term and definition, identify if they correspond to a functional pair, that is, if the term obtained corresponds to the definition; if this does not happen, turn one of the two cards over and pick up another, in order to discover if it matches a functional pair or not. It should be noted that, in this task, mathematical connections of meaning and feature are promoted (**Figure 9**).

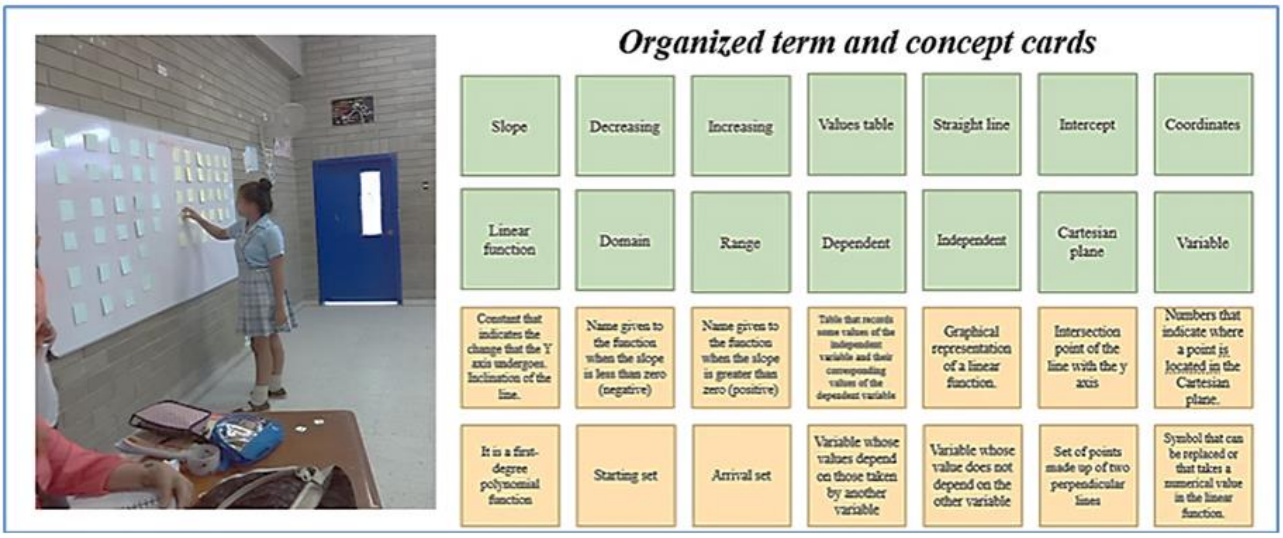


Figure 9. Functional couple game (Source: Field study)

1. You are the owner of a company, and you pay a worker $24,000 per day worked. How much should you pay the worker for 3 days worked? How much should you pay him if he works 6 days, 9 days, 12 days, 15 days, 18 days, 21 days, 24 days, or a month?
2. For a marbles game, each participant is assigned 4 marbles. If 3 participants enter the game, how many marbles are available for the game? If there are 5, 7, 9, 11, 13, 15 participants, how many marbles are assigned to each game respectively?
3. Darío opened a savings account in the Bogotá bank with $45,000 and deposits $35,000 every week. How much money should you have in your account after a week? As time passes, how much money will Dario have after 2, 3, 4, 5, 6, 7, 8, 9 weeks?
4. 4. In a nursery school, each child has 2 palettes to paint with tempera paints. After 1 hour, the teacher gives him 3 more palettes to paint. How many painted palettes will each child have after 1 hour? Later, how many painted palettes will each child have after 2, 3, 6, 10, 15, 18, 21, 25 hours?
5. 5. A lender lends at 10% monthly. Santiago lends him $170,000. How much will Santiago have to pay the lender after 1 month? How much will he have to pay after 2 months, 3 months, 4 months, 5 months, 6 months, 8 months, 10 months, 1 year?
6. The owner of a door painting company pays each worker a base salary of $350,000 plus $8000 for each door painted. How much should a worker charge for 2 doors, 3 doors, 4 doors, 5 doors, 6 doors, 7 doors, 8 doors, 9 doors, 10 doors?

Figure 10. Problems and manipulative materials used in variation situations I (Source: Field study)

VT 6: Variation situations

Students are organized into pairs to promote discussion and reflection between them in solving problem situations. Each pair is given an instruction manual with six problem situations (in order to encourage modeling connections), whose demands are oriented toward the analysis of the situation posed through the construction of a table of values and the graph on the Cartesian plane (promoting the connection of different types of representations). For the development of the mathematics laboratory as a means of simulating variation situations and solving the problem situations posed, manipulable materials are used: non-commercial bills and coins in Colombian pesos, marbles and popsicle sticks (**Figure 10**). These materials were given equally to each pair of students depending on the assigned situation. In addition, they are given support material, a cardboard card for the tabular representation and a ruler with millimeter sheets for the construction of the graph on the Cartesian plane.

VT 7. Application and generalization problems on the linear function

In the development of this activity, aspects of the previous activity were considered, such as the distribution and organization of the students and the use of manipulable materials (non-commercial bills). Each pair of students is given the instructions with five problem situations whose demand consists of the construction of a numerical and tabular representation, in addition to the algebraic representation of the situation as a form of generalization of the situation posed (modelling connections and different representations must be activated). The objective is for the students to identify the linear behavior of the situation through the regularity analyzed and to represent it algebraically. Based on this analysis and representation, construct the graphic representation on the millimeter sheets. In addition, analyze at what moment this representation is increasing or decreasing (**Figure 11**).

1. Triple A charges $9,000 for water service as a fixed charge plus $1,000 for each cubic meter of consumption. In Carolina's house they consumed 25 cubic meters. What is the total value to be paid?
2. A group of friends must pay $49,000 to see the movie "Anabelle." At the theater, a combo of a soda and a large popsicle costs $16,000. If they decide to buy 9 combos, how much should they pay in total including tickets?
3. Tigo mobile telephony offers a postpaid plan that includes 2Gb internet and minutes to any destination. For the monthly consumption of the service, $100 is charged for each minute consumed and $32,000 for the entire internet package. If Manuel purchased the plan, consumed 167 minutes and the entire internet package in the month, how much should Manuel pay on the bill?
4. In Nacho's house there is a tank with 40 pounds of rice. 1.5 pounds of rice is consumed daily. How many pounds of rice are in the tank after 22 days?
5. The bus ticket costs $1600 per person. If 487 people have ridden in one day, what was the income that the owner of the bus earned on that day?

Figure 11. Tasks and manipulative materials used in variation situations II (Source: Field study)

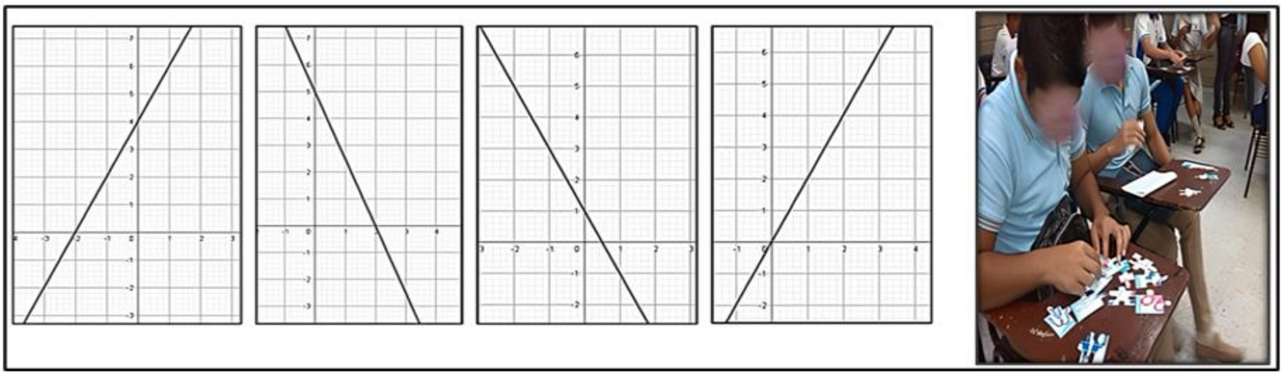


Figure 12. Puzzle from the graphical to algebraic representation of a linear function (Source: Field study)

VT 8. From graphical representation to algebraic representation

Previously, instructors explained to students how to perform the analysis of the graphical representation of a linear function, which includes the analysis of the behavior of the independent variable with respect to the dependent variable, whose values are recorded in the table of values and from this, the generalization of the linear function is built. To do this, they rely on posters representing this process and promote familiarization with the representations of situations of variation and change in the development of variational thinking with the linear function (mathematical connections of different types of representations). The task consists of students putting together a puzzle containing the graphical representation of a linear function (see **Figure 12**), based on this representation, building a table of values, analyzing the variation and building the algebraic representation of said function from the regular behavior identified (connections of different representations), supported by the materials from the mathematics laboratory. Furthermore, in this task the activation of the part-whole connection is evident because they are considering particular cases or pieces of more general graphs. That is, there are particular-general relationships.

Final Test

This test was carried out at the end of the implementation of the mathematics laboratory to compare the results with the diagnostic test (**Figure 13**). The comparison allows us to analyze the initial and final situation of the students in relation to variational thinking in the linear function. It was also based on questions about the concept of function, concepts and meanings related to the linear function (connection of meaning), modeling situations and graphical representation of the linear function (modelling connections, part-whole and different representations).

DATA ANALYSIS

To analyze the data, the following were considered:

(a) review and organize the written productions corresponding to the three phases of the research (diagnostic test, variational laboratory tasks and the final test),

(b) establish categories and subcategories of analysis (**Table 4**), and

(c) analyze the data considering the role of mathematical connections.

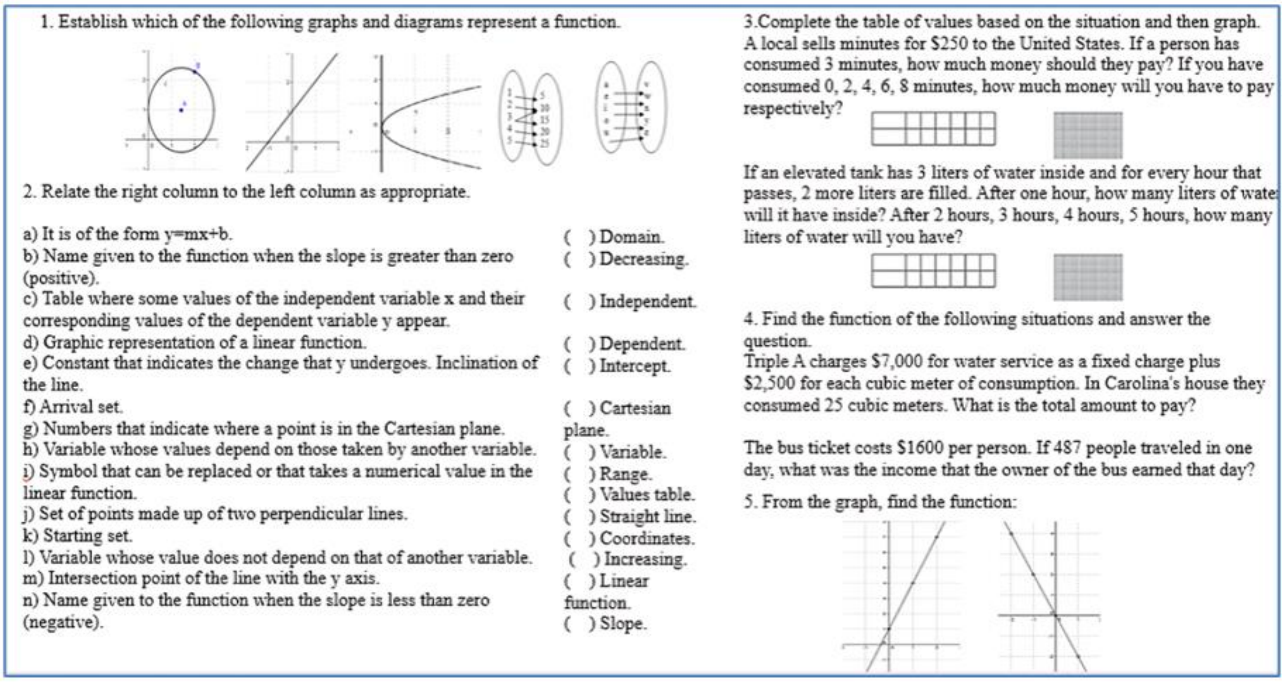


Figure 13. Final test applied to students (Source: Authors' own elaboration)

Table 4. Categories and subcategories of analysis

Categories	Subcategories: Concepts addressed in the tasks
Diagnostic test	Generalization of patterns
	Location and graphical representation on the Cartesian plane
	Problem-solving situations involving variation
Mathematics laboratory	Function concept
	Concept of linear function
	Resolution of problem situations involving variation with linear functions
	Representation of graph to algebra
Final test	Function concept
	Resolution of problem situations involving variation with linear functions
	Representation of graph to algebra

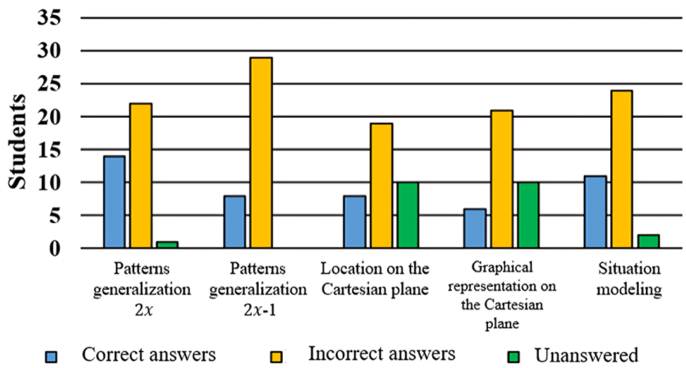


Figure 14. Diagnostic test results (Source: Authors' own elaboration)

Actually, the categories and subcategories refer to an organization of the themes associated with the tasks and in this way they are presented in the following.

RESULTS

Diagnostic Test

The results obtained in the diagnostic test are presented (**Figure 14**). The evidence shows that the students had difficulties in mastering concepts associated with the linear function and in modeling situations that involve the generalization of patterns and multiple representations because they did not establish the mathematical connections necessary to solve the VTs of initial test.

Figure 15. Students assume the graph of the function as a path by traveling on it (Source: Field study)

In the linear pattern generalization ($2x$), 62.16% of students answered incorrectly or did not answer, while 37.83% answered correctly. In the $2x - 1$ form generalization, 78.37% answered incorrectly or did not answer, while 21.62% answered correctly. Students who answered incorrectly showed difficulties in a) working with distant particular cases (mathematical part-whole connection), b) recognizing regularity, and c) constructing the general rule in relation to the proposed linear functions.

In the location on the Cartesian plane, 51.35% of the students located the coordinates incorrectly or did not respond and 21.62% located them correctly by establishing connections of alternative representations. In the graphic representation, 56.75% of the students did it incorrectly while 16.21% represented them correctly, for example, this occurs in the location, most of the students do not know how to locate coordinates on the Cartesian plane, mainly because they confuse the abscissa axis (x) with the ordinate axis (y) and vice versa. Likewise, they have difficulties in performing horizontal and vertical movements on the Cartesian plane. In relation to the graphical representation, there were students who located the coordinates correctly, but did not know how to represent them graphically, which is consistent with what has been reported by some researchers on the problem of understanding functions (Díaz, 2013; Fiallo & Parada, 2014; Ordóñez-Ortega et al., 2019).

In modelling problem situations, 64.86% of students did not show variational thinking, while 29.72% did. This phenomenon occurs because most students found it difficult to model situations variationally from magnitudes and represent them through numbering, tabulation, algebraic symbolization and graphing. In other words, students need to learn to establish modeling connections linked to the understanding of the problem situation and the multiple representations of the mathematical concepts involved in it.

Implementation of the Mathematics Laboratory

Function concept

Based on the results obtained in the diagnostic test, the mathematics laboratory began with a more in-depth study of the concept of function. To do so, three VTs were developed:

(1) the human Cartesian plane,

(2) relates and functions, and

(3) coordinates on the Cartesian plane.

The first two were developed in a group and explanatory manner, while the third was developed individually.

In the first task, the results obtained were satisfactory, because most of the students learned to locate coordinates on the Cartesian plane correctly through their bodies and recognized when a graph represents a function or not, through the vertical line test. The graphic representations they built with their bodies were: a parabola with a vertex on the ordinate axis, straight lines (increasing, decreasing, and constant), a circle and a square. This procedure shows the metaphorical connection where the students locate themselves considering that the graph is a straight line when they are one behind the other, assuming that each of the students was a point of Cartesian coordinates and they simulated a path (**Figure 15**).

In the second activity, students understood that in a function, the elements of the starting set (domain) are related to the elements of the arrival set (range) and only with one, otherwise it only represents a relationship (connection of a feature type). The manipulative materials of this activity allowed students to distinguish, from the visualization, a relationship of a function. Later, students made proposals with the materials about connections of different representations (equivalent and alternate) distinguishing a relationship of a function $y = f(x)$ by means of a sagittal diagram (**Figure 16**).

In the third task, 21.6% of the students had difficulties caused by the lack of connections in the location and representation on the Cartesian plane and 78.4% did it correctly. The difficulties in the location of coordinates were associated with the erroneous uniform distribution of the grids (scales) on the board provided, the incorrect use of the ruler as a measuring instrument and the distinction between the coordinate axes. It is recognized that the materials used in the mathematics laboratory in this activity allowed the students to strengthen the location and graphic representation on the Cartesian plane (**Figure 17**).

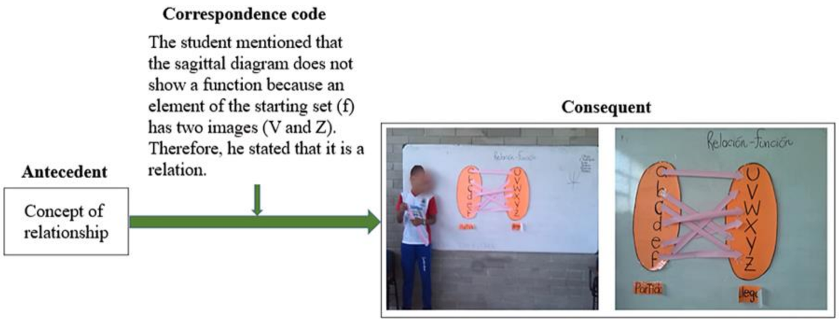


Figure 16. Representation of the relationship by means of the sagittal diagram (Source: Field study)

Figure 17. Graphical representations on the Cartesian plane (Source: Field study)

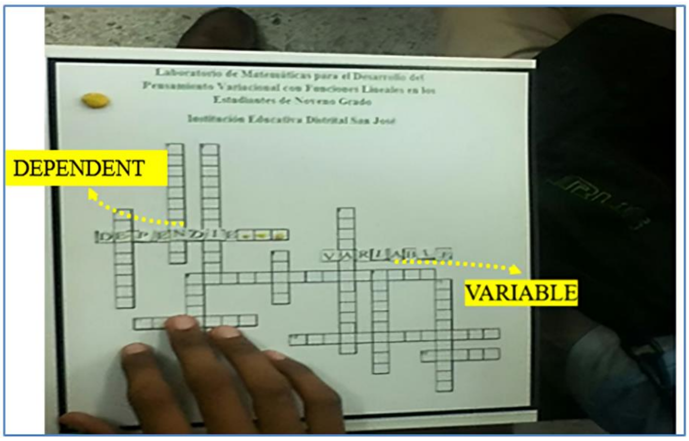


Figure 18. Using crossword puzzles to reinforce the meanings of concepts (Source: Field study)

Concept of linear function

In the diagnostic test, it was identified that the students had difficulty in relation to the mastery of the functional and variational language of the linear function, one of the causes is due to the fact that the students had not previously had contact with these concepts and the previous ones. For this purpose, the proposed activities consisted of a crossword puzzle with terms related to the linear function and the find your functional partner activity. In the crossword puzzle activity, the students were interested in participating and showed mastery of the concepts related to the linear function, activating the connection of meaning (**Figure 18**). Most of the students completed the crossword puzzle correctly.

The functional pair activity was carried out in groups. During the activity, most students demonstrated mastery of the concepts of linear function and were participative, focused, motivated and analyzing their answers (**Figure 19**).

Resolution of problem situations involving variation with linear functions

The students were organized into 19 pairs. The VT demand in this session consisted of the construction of tabular and graphical representations for each variational situation. In the tabular representation, 89.47% of the students correctly constructed the double-entry table of values in relation to the variables considered in the tasks and 10.52% did so incorrectly (**Figure 20**). Most of the students relied on the manipulative materials to model the situations and in the construction of the tables they showed analysis, reasoning and interpretation of the variables involved, reflecting on how the values of the independent and dependent variables were increasing or decreasing in each situation. In addition, they reflected on the relationship between both variables.

Figure 19. Establishing functional pairs (Source: Field study)

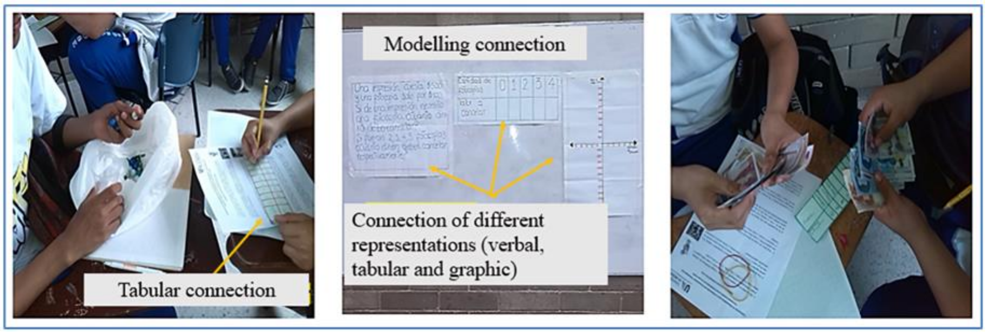


Figure 20. Connecting different representations to solve problems (Source: Field study)

This type of representation supported the development of variational thinking in most of the students and in turn, the connections between different representations (**Figure 20**).

Regarding the graphical representation of the linear function, the same results were presented in accordance with the previous representation, and it was the same students who correctly demonstrated the construction of the graphical representation. Based on the tables of values, the students constructed the graph of the function they represented (conversion from one system to another–alternative representations) and based on this representation (straight line), the students identified that it corresponded to a linear function.

Application and generalization of the linear function

The students were organized into nineteen pairs. The VT demand in this session consisted of the construction of tabular, algebraic and graphical representations for each variational situation. In the tabular representation, 94.73% of the students correctly constructed the tables of values of the linear functions and 5.26% did so incorrectly. Unlike the previous task, in this one they were not provided with particular or specific cases of the situation. For this, the students, based on their analysis of the situation, constructed it to record the behavior of the linear function analyzed. To do this, they involved arithmetic operations to complete the table of values, and they decided which values of the independent variable they would record in the representation and find the values of the dependent variable.

In the algebraic representation, 94.73% of the students correctly constructed the algebraic expressions of the linear functions and 5.26% did so incorrectly. For the construction of these expressions, the students relied on the tabular representation, based on the analysis of the regularity behavior between the particular cases they chose (part-whole connection), they represented this behavior through letters (variables) (conversion from one system to another). It is recognized that the students have generalized due to this process of identifying regularity and constructing the general rule through the algebraic expression of a linear function.

In the graphical representation, 84.21% of the students correctly constructed the graphical representations of the linear functions and 15.79% did so incorrectly (**Figure 21**). Based on the algebraic representation, the students recognized what type of graphical representation they had to construct and with the values recorded in the value tables, they proceeded to locate them as coordinates on the Cartesian plane to finally represent the linear functions.

From graphical representation to algebraic representation

The students were organized into nineteen groups of two students. In the algebraic graphical representation, 84.21% of the students did it correctly and 15.78% did it incorrectly (**Figure 21**). In this task, most of the students analyzed the behavior of the function through the correspondence pattern in relation to the independent and dependent variables, which allowed them to represent it algebraically. Some students analyzed the value of the slope at two points on the line and based on the intercept with

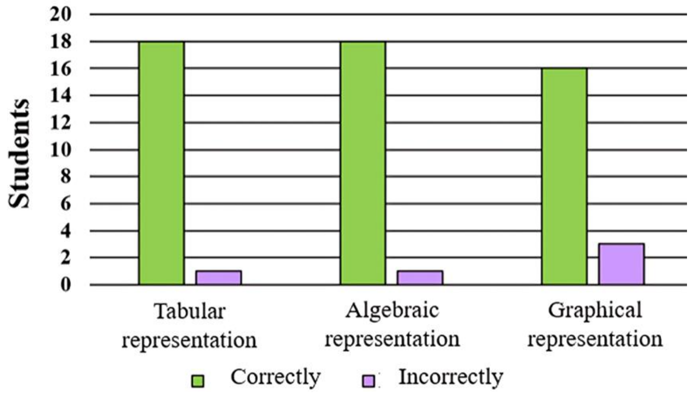


Figure 21. Results of the resolution of variational situations II (Source: Authors' own elaboration)

Figure 22. Linear function puzzles (Source: Field study)

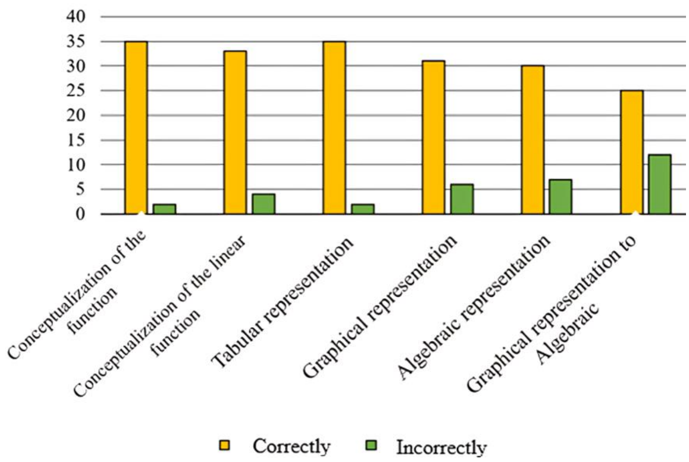


Figure 23. Final test results (Source: Authors' own elaboration)

the ordinate axis, they represented the function in the form $y = mx + b$. The linear functions represented algebraically were: $y = 2x + 4$; $y = -\frac{5}{2}x + 5$; $y = -2x + 1$; $y = 2x$, which shows the procedural connections and different alternate representations. The students were motivated to do the task because they had previously assembled the puzzle that contained the graphical representation (**Figure 22**).

Final test

In this test the results were satisfactory due to the evidence of variational thinking in the students through the linear function and the established connections (**Figure 23**).

In the conceptualization of functions, 94.59% of the students answered correctly and 5.4% incorrectly (**Figure 23**). The students showed progress in the conceptualization of the function compared to the diagnostic test. In the conceptualization of the linear function, 89.19% of the students answered correctly and 10.81% did not (**Figure 23**). The students showed that through the tasks carried out in the mathematics laboratory they strengthened the concepts related to the linear function and it is important in the mastery of mathematical concepts.

In the tabular representation, 94.59% of the students constructed these representations correctly while 5.4% did so incorrectly; in the graphical representation, 83.79% of the students did so correctly and 16.21% did not do so correctly; in the algebraic representation, 81.08% constructed algebraic expressions correctly while 18.91% did so incorrectly (**Figure 23**). The results in the construction of multiple representations in the development of variational thinking in relation to the linear function show that the students made significant progress and most of them strengthened their variational skills and the analysis, reasoning and variational interpretation of the linear function in different registers was promoted. In addition, 67.57% of the students correctly converted from the graphical to the algebraic representation and 32.43% incorrectly (**Figure 23**).

DISCUSSION

This research presented concrete examples of the development of variational thinking in high school students through a mathematics laboratory with linear functions. During the implementation of the laboratory, the mastery of variational language was strengthened in relation to the linear function, specifically with the analysis of problem situations that will involve the construction and analysis of multiple representations, supported by manipulative materials. Vasco (2006) emphasizes that variational thinking is linked to the study and conversion of different representations related to pattern recognition, variation and construction of generalizations. In fact, the students went through the multiple representations of the function under study, since emphasis was placed on graphing functions in the Cartesian plane and establishing mathematical connections with sagittal diagrams that account for the domain, range and codomain.

Furthermore, this study shows relevance regarding the current research problem on the difficulties in understanding functions by students and some teachers, given that most authors (e.g., Esquer-Armenta & Félix-Romero, 2021; García-García, 2024; Hansen & Naalsund, 2022; Hrnjičić & Alihodžić, 2024; Moss et al., 2020; Ordóñez-Ortega et al., 2019) state that there are difficulties in identifying the function and distinguishing it from a relation, constructing and manipulating the records of representation of the functions, lack of knowledge of the concept and use of variables (dependent and independent), solving problems with functions that involve variational thinking, among others. In fact, although various activities are not addressed to solve the problems evidenced, the proposed VTs are an input that allowed to visualize and detail the multiplicity of representations of the functions and their application. For this reason, it is asserted that the implementation of this didactic proposal is aimed at what is suggested by the school mathematics curriculum (MEN, 1998) and research regarding variational thinking, likewise, environments that facilitate the interaction and exploration of variational contexts are promoted.

The discussion on variational thinking and linear functions reflects the importance of establishing connections between different representations (algebraic, graphical and tabular), and how these connections are essential for the development of more advanced mathematical skills. As Núñez and Correa (2019) and the MEN (1998) point out, variational thinking does not develop in isolation, but through the interaction between different mathematical reasonings. However, one of the main difficulties for students is precisely establishing these connections, especially when it comes to functions and variation between variables (García-García, 2024).

Linear functions, represented by the equation $y = mx + b$, are a key introduction to the concept of variation, as they describe a relationship of constant change between variables. The slope m of the line allows us to visualize how one variable depends on the other, and its graph on the Cartesian plane provides a clear visual representation of this relationship. However, despite its simplicity, many students face difficulties in connecting algebraic representations with graphs and tables of values, which limits their ability to understand and solve problems related to variation (Esquer-Armenta & Félix-Romero, 2021). In this context, the activities of the linear functions laboratory play a fundamental role in overcoming these difficulties. As observed in the laboratory described previously, activities such as locating points on the Cartesian plane and using manipulative materials allowed students to practically experience the relationship between variables, which helped them understand the function in a more tangible and visual way. In particular, the task of representing functions using the human Cartesian plane (where students line up as points on the graph) proved effective in showing how a linear equation generates a clear graphical representation of the variation between variables.

The transition between different representations, facilitated by the use of tools such as manipulatives and technologies, is key for students to be able to connect algebraic, graphical, and tabular representations in a coherent way. The laboratory experience demonstrated that students, by constructing and analyzing tables of values and graphs, were able to reflect on the relationships of variation and how the values of one variable depend on the other, which is fundamental for the development of variational thinking. However, as Díaz (2013) points out, many students fail to identify the purpose of the variables or correctly interpret graphical and algebraic representations due to the lack of a solid understanding of the connections between them. In this sense, the results of the laboratory show that, despite the progress, some students still have difficulties, such as in the construction of graphical representations or the identification of patterns of variation. This underlines the importance of adopting pedagogical approaches that promote active exploration and the use of interactive technologies to strengthen these connections (García-García, 2024).

Finally, the implementation of pedagogical strategies that promote active learning and the manipulation of representations is crucial to consolidate variational thinking. Through an approach based on mathematical modeling of real phenomena, such as situations of constant variation or change, students can deepen their understanding of linear functions and also of more complex mathematical functions. The linear functions laboratory not only allowed students to understand functions as relationships of constant variation, but also provided them with the tools to think about and solve mathematical problems in more dynamic and real contexts.

CONCLUSIONS

We maintain that this work contributes to the lines of research on mathematical connections and variational thinking, but it could be further developed by using these theoretical approaches in an articulated way for the analysis of phenomena in mathematics education or favoring transversality with different subjects (biology, chemistry, physics, etc.) where functions, derivatives and variations are applied. From other angles, we assure that this work, as well as having strengths, also presents limitations because only manipulative teaching materials were used and one of the contexts that favors the understanding of functions is the technological one through software such as GeoGebra that allows establishing algebraic-graphic-symbolic connections simultaneously.

It is emphasized that it is important to integrate pedagogical approaches that favor connections between different representations, procedures, patterns, etc. to develop variational thinking. Laboratory activities and the use of interactive technologies are valuable resources that, by promoting the manipulation of representations with concrete material and reflection on variation relationships, can help students overcome common difficulties they face when learning about functions from their variation and change.

Author contributions: KN-G: conceived the idea of the research and wrote the article and collected the data; **CAR-N:** helped write the article and reviewed the data analysis; **LC-S:** contributed to the literature review and construction of an instrument for data collection; **VFM:** improved the framework and the final version of the manuscript. All authors have agreed with the results and conclusions.

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Ethical statement: The authors stated that the participants and the educational institution were informed that this article has educational purposes and not economic or political ones and agreed to participate voluntarily. In addition, one of the researchers was a full-time professor at the institution with research commitments and is authorized to conduct research with the student population with the purpose of improving students’ understanding of mathematical concepts. Written informed consents were obtained from the participants or their legal guardians.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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