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Diagnostic study of mathematical reasoning in novice university students

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ABSTRACT

This paper presents the results of a study aimed at diagnosing how newly enrolled mathematics majors reason from a mathematical standpoint when they are required to perform inductive and deductive processes and validate or refute claims of a general nature. The theoretical framework underpinning this research comprises studies and investigations related to mathematical reasoning. The results of this diagnostic reveal that students lack mathematical reasoning skills that would allow them to justify statements of a general nature. In addition, they face difficulties in the development of deductive reasoning, are prone to generalize based on specific cases and lack skills in mathematical argumentation. On the other hand, they lack basic algebra skills such as developing square binomials and factoring. These findings highlight the need to design and implement a didactic proposal that fosters the development of reasoning and argumentation in students.

Keywords: mathematical reasoning, deductive reasoning, inductive reasoning, justification

INTRODUCTION

Mathematical reasoning is a fundamental skill for learning and teaching mathematics. Thus, several researchers have highlighted the role that mathematical reasoning plays in the construction of knowledge and discovery in mathematics (Hadamard, 1945; Lakatos, 1978; Polya, 1966, 1967). In particular, Polya (1966) claims that mathematical results are ensured or established by deductive reasoning, while inductive reasoning (as a part of plausible reasoning) plays an important role in the formulation of conjectures (mathematical discovery). In this context, research in the field of mathematical education points out that it is essential to promote the development of mathematical reasoning and proof in the mathematical education of students at all levels (Arsac, 1996; Arsac & Mante, 1997; Balacheff, 1987; Bravo et al., 2001; Crespo, 2005; Duval, 1992; Flores, 2007; Larios, 2003, 2019; Marmolejo & Moreno, 2019). By their own account, Mata-Pereira and da Ponte (2017) contend that mathematical reasoning not only provides tools to prove but also leads to a deeper understanding of mathematical concepts.

On the other hand, in international documents of reference, such as the various editions of "principles and standards for school mathematics" (NCTM, 2000), reasoning and proof are established as one of the five standards of processes with which students must become familiar and improve, facing increasing complexity as they advance in their education. In the framework of PISA, the Organization for Economic Cooperation and Development (OECD, 2017) highlights reasoning and argumentation as fundamental mathematical skills. This implies processing thoughts in a logical way, making inferences, and presenting claims that substantiate the solution to a problem.

In accordance with the above, Jeannotte and Kieran (2017) assert that one of the primary objectives of study plans should be to foster mathematical reasoning in students. This is reflected in the mathematics programs of almost all countries, where crucial importance is given to activities aimed at its development.

In Mexico, the plans and study programs overseen by the Secretaría de Educación Pública [Secretary of Public Education] (SEP, 2011, 2017a, 2017b) from basic education (BE) to high school education (HSE) emphasize the importance of the development of mathematical reasoning. Starting the preschool, the aim is for students to apply mathematical reasoning in situations involving counting and first numbers. In secondary education, it is established that deductive reasoning should be promoted by working with properties of geometric figures while also reinforcing inductive and analogical reasoning. When BE is concluded, it is expected that students can apply mathematical reasoning in diverse situations. In HSE, plans and programs of study highlight the importance for students suitable to argue and structure their ideas, judgments and reasonings.

However, Codina and Lupiañez (1999) point out that in elementary school classrooms, the development of mathematical reasoning and proof in students does not have the primary role recognized by research in the field of teaching and international documents that establish standards for school mathematics. Teaching at these levels focuses more on the knowledge of mathematical content and the treatment of algorithmic procedures.

The disdain for mathematical proof and reasoning at elementary school levels produces in students' severe difficulties when entering some college mathematics majors (Di Martino & Gregorio, 2019). Tall (2008) investigated the changes in thinking involved in the transition from school mathematics to formal demonstration in college mathematics majors. In his study, he highlights three mutually supportive mathematical worlds: conceptual-embodied, proceptual-symbolic and axiomatic-formal. The first, based on the experience of objects with the real world. The second, corresponding to algebraic calculations and manipulations and includes imaginary concepts (e.g., numbers). The third, concerning definitions and demonstrations. Likewise, Tall (2008) states that pre-university mathematics is situated both in the conceptual-corporate world and in the proceptual-symbolic world, while university mathematics corresponds to the axiomatic-formal world. In other words, the fullest deployment of mathematical reasoning and demonstration occurs at the university level. At this level, demonstration is an integral part of the mathematical methodology in mathematics majors.

In the context of mathematics, there exists a great diversity of elements impacting the pass from teaching in pre-major education to major education (Gómez-Chacón, 2009). Some of these highlighted elements include disparities between educational levels, a lack of skills for mathematical reasoning and problem-solving, which limits the correct use of formal language, reasoning with rigor, or detecting logical inconsistencies (Huidobro et al., 2010). More precisely, Geisler and Rolka (2021), Jiménez and Areizaga (2001), Larrazolo et al. (2013), Liebendörfer and Schukajlow (2017), Posso et al. (2007), Rach and Heinze (2016), and Sánchez-Matamoros et al. (2022) agree that the transition from high school to university involves a shift from "expository" mathematics (based on memorization, procedures, and exemplification) to "superior" mathematics (emphasizing formal reasoning). In particular, Larrazolo et al. (2013) confirmed a low level of mathematical reasoning skills among high school graduates in Mexico. They emphasize that mathematical reasoning is a competence that should have theoretically developed during BE.

Despite the highlighted importance in official documents at national and international levels about the foster of mathematical reasoning from the prescholar stage until high school, reality in classrooms contradicts this directive. There is a marked discrepancy between theory and practice in the development of mathematical reasoning. Although educational literature and curriculum documents acknowledge its importance in contributing to knowledge construction and critical thinking, studies reveal that at lower educational levels, there is a prioritization of memorization and procedural mechanization without delving into the reasoning behind them. This gap between what is established in the study plans and the reality in the classroom generates a disconnection between both educational levels.

Consequently, students advance in their education without having adequately developed their reasoning, which causes them difficulties when reaching more advanced educational levels, such as, for example, in a mathematics degree.

The reported literature and the study plans at the elementary and high school levels allowed us to recognize the importance of promoting the development of mathematical reasoning. In this context, the aim of the research was to examine how first-year students' reason mathematically when performing inductive and deductive procedures, and to validate or refute general statements.

The results obtained in this diagnosis will constitute the essential starting point for designing subsequent activities.

From the information here presented, the research questions in this paper are the following:

How do first year mathematics students validate or refute general statements? Are they able to perform simple processes of induction and deduction?

THEORETICAL FRAMEWORK

Reasoning and Reasoning in Mathematics Education

According to Brodie (2010), reasoning involves developing lines of thought or arguments that can be useful for various purposes, either to establish one's own conviction or to convince others of the claims being made. It also serves to solve a problem or to integrate a series of ideas into a single coherent one. Furthermore, according to this author, mathematical reasoning develops upon and with mathematical objects. This involves a greater effort on the part of the individual to understand and approach a mathematical problem, process the available information and construct new one.

Balacheff (1987), highlights that thinking and validation processes are addressed through the study of proof process. He considers reasoning as an intellectual activity through which new information is modified and produced. According to Balacheff (1987), an explanation is an argument, which in turn, is a form of reasoning. Furthermore, emphasizes the use given to "proof" and "proving" as synonyms in mathematics education. For this author, proof (*preuve*) is a justification whose validity can be modified according to the experience of the community and time. Mathematical proof, on the other hand, is specific and logical; its structure is strongly based on a set of definitions, theorems and valid rules shared by the mathematical community. Duval (1992) disagrees with Balacheff's (1987) assertion and explains the differences and functions of argumentation, explanation, reasoning, and proof. Duval (1992) states that explanation goes hand in hand with argumentation and defines it as follows: an explanation gives one or more reasons to make a datum (a phenomenon, result, behavior, etc.) comprehensible. Another aspect to mention is the difference between argumentation and proof. According to Duval (1992), these are distinct concepts. The separation between them depends on organizational links. Duval (1992) defines reasoning, as follows:

> An organization of propositions that is directed towards an objective statement in order to modify the epistemic value that this objective statement has in a given state of knowledge, or in a given social environment, and as a consequence, modifies its truth value when certain particular organizational conditions are fulfilled (p. 52).

Duval (1992) adds that reasonings considered as proofs must to be valid reasonings; the opposite occurs with argumentation, which does not follow validity links but rather pertinence links. In addition, reasoning requires explanations; it is linked to the production of acceptable reasons or arguments. Moreover, the transition from argumentation to reasoning implies going beyond the discussion that can be generated in a group or the internalization of such discussion (Duval, 1992). In 2019, Balacheff (1987) recognizes that in 1987, he had adopted an inappropriate conception of reasoning, oriented more toward the psychological field. Therefore, in his paper he emphasizes the viability of revisiting Duval's (1992) position, which is adequate and congruent with the theoretical frameworks on which Balacheff (1987) relies.

For O'Daffer and Thornquist (1993) mathematical reasoning is part of mathematical thinking, which involves forming generalizations, drawing valid conclusions about ideas and explaining their relationships. From the point of view of Arsac and Mante (1997), mathematical reasoning allows for validating or refuting results in both the scientific and academic fields.

According to Arsac (1996), reasoning aims at discovering new knowledge by examining what is already known, designating it as the intellectual activity that leads to the intended goal, expressed as the result verbalized orally or in writing. From their perspective, Brousseau and Gibel (2005) define reasoning from a formal language, as follows:

> A relation R between two elements A and B such that; A denotes a condition or a fact, which could depend on particular circumstances; B is a consequence, a decision or a predicted fact; R is a relation, a rule or, in general, something that is considered known and accepted. The relation R leads the acting subject (the reasoning "agent"), in the case that the condition A is fulfilled, or the fact A occurs, to make the decision B, to predict B or to affirm that B is true (p. 17).

Other remarkable works in this line of study on mathematical reasoning include those reported by Arsac and Mante (1997), Benítez et al. (2016), Jeannotte (2015), Jeannotte and Kieran (2017), Mata-Pereira and da Ponte (2017), Niswah and Qohar (2020), Pedreros (2016), Saorín et al. (2019), and Torregosa et al. (2010). These studies highlight essential elements such as the treatment of justified inferences, argumentation, classification, comparison, identification of patterns, generalization, conjecturing, justification, proof and exemplification. Additionally, they explore the logical sense of a statement, the meaning of statements of a general nature and strategies to refute statements using counterexamples.

As this paper shows, there is a considerable amount of research in the mathematics education field, focused on the study of mathematical reasoning. It is observed that when comparing these research studies, there are more similarities than differences, and the latter are not significant enough to warrant exploring any other alternative stance. Instead, the exchange of ideas enables us to establish a stance on mathematical reasoning in this work, which will be further developed later.

Types of Reasoning

In the literature on mathematics education, three main types of reasoning are distinguished: deductive, inductive, and abductive.

According to Fabert and Grenier (2011), it is common in teaching to use deductive reasoning, which is employed for proving and justifying. This involves starting from properties recognized as true, by establishing a chain of logical sentences that leads to a specific property. Flores (2007) also states that deductive reasoning involves generating chains of ideas or reasons that lead to a conclusion, which must be supported by one or more premises.

Polya (1966) argues that inductive reasoning usually begins with an observation. Then, a pattern of behavior is identified, allowing for the generalization and formulation of a conjecture. An attempt will be made to discover whether it is true or false. It is evident that the induction performed may lead to incorrect results. However, it is crucial not to forget that, although it is possible to be wrong, on many occasions the induction can be correct, but the results thus obtained in mathematics have to be validated through deductive reasoning. In this sense, it is also important to mention the fact that, if a result is true for a large number of cases, this is not sufficient for its validation. However, if such a general result or statement is not satisfied, even for a single case, the statement is invalidated. A case that refutes the statement is termed a counterexample. According to Gómez-Chacón (2009), inductive reasoning is used to discover general laws from the observation of particular cases and their combinations. Its process consists of moving from the particular to the general, where generalization is linked in many cases to another previous process of particularization. By means of observation, induction tries to discover regularity and coherence, employing generalization, particularization and analogy as its most visible instruments.

Castro et al. (2010) point out that, both in the scientific and social fields, inductive reasoning is a powerful tool for the construction of knowledge thanks to the fact that generalization is one of its fundamental components. In this sense, the researchers state that the following seven steps favor the process of inductive reasoning:

1. **Work with particular cases:** Specific cases or examples with which the process is initiated. They are usually simple and easily observable cases.
2. **Organization of particular cases:** Arranging the data obtained in a way that helps the perception of patterns, either in a table, rows and columns, with some order.

3. **Pattern identification:** The pattern, or guideline, is that which is common, that which is repeated regularly in different events or situations, and which is expected to recur.
4. **Formulation of conjectures:** A conjecture is a proposition that is suspected to be true but has not been subjected to exploration. Such exploration may result in its acceptance or rejection.
5. **Justification of conjecture:** Refers to any reason given to convince of the truth of an assertion. They usually distinguish between empirical and deductive justifications. The empirical ones use examples as elements of conviction. It is checked again with other particular cases.
6. **Generalization:** The conjecture is expressed in such a way that it refers to all cases of a given class. It implies the extension of reasoning beyond the particular cases considered.
7. **Proof:** Formal validation process in which there is no doubt about the validity of the conjecture to be proved and determines it unequivocally.

For Soler-Álvarez and Manrique (2014), the types of reasoning that contribute to the process of mathematical discovery in the classroom are:

- Abductive (has implicit the idea of creativity, supports the formulation of conjectures).
- Inductive and deductive (support the verification and validation of conjectures).

In the framework of this research, we assume that mathematical reasoning is a logical and intellectual thinking process or activity used by every individual to create new ideas (Balacheff, 1987), solve problems (Brodie, 2010) and most of the time it is implicit (Benítez et al., 2016). Its objective is to discover new knowledge through the examination of what is already known (Arsac, 1996). It is part of mathematical thinking (O'Daffer & Thornquist, 1993), its study is linked to argumentation (Godino & Recio, 2001) and is shared in the collective argumentation process (Conner et al., 2014). Reasoning allows new information to be obtained from previous information (Saorín et al., 2019; Torregosa et al., 2010). It is related to other processes, such as generalizing, conjecturing, identifying a pattern, comparing, classifying, justifying, proving, demonstrating, and exemplifying (Jeannotte & Kieran, 2017). The types of reasoning that contribute to the mathematical discovery process in the classroom are abductive, inductive and deductive (Soler-Álvarez & Manrique, 2014). Abductive reasoning and inductive reasoning allow us to generate conjectures of a universal or general nature, but only deductive reasoning allows us to validate them. There are three rules of great importance for mathematical reasoning:

(1) a statement is either true or false, but not both,

(2) a "large number of cases" do not prove the validity of a general statement, and

(3) a single counterexample is sufficient to disprove a statement (Arsac & Mante, 1997).

METHOD

This research is qualitative and exploratory, since we were interested in diagnosing how students' reason mathematically when they carry out inductive and deductive procedures, validating and refuting general statements.

Context and Participants

The study was carried out with newly enrolled mathematics majors. The degree provides training in mathematics in five areas (pure mathematics, educational mathematics, applied mathematics, statistics and computer science). Students in all areas must take a set of common subjects to ensure a solid mathematical background to enable them to address the content area of their choice. The peculiarity of the students who enter this degree program is that most of them come from rural communities where living conditions are precarious. Therefore, during their high school studies they faced a difficult academic situation, marked by the lack of resources, limited teacher training and the limited supply of educational materials, among other factors. The diagnostic questionnaire was applied in the first week of the 2022-2023 school year (final week of August 2022) and 44 students divided into two groups participated.

Diagnostic Questionnaire

A diagnostic questionnaire was designed with three activities that students had to answer individually. The first consisted of statements that students had to evaluate as true or false and then give as rigorous a justification as possible for their answers (see **Figure 1**).

Activity 1: State whether the following statements are true or false. Give as rigorous a justification as possible for your answers.

a. The sum of two multiples of 7 is a multiple of 7.
b. If n is an odd number, then $n^2 - 1$ is a multiple of 8.
c. For each integer number n, the integer $n^2 - n + 11$ is a prime number.

Figure 1. Activity 1 of the diagnostic questionnaire (Source: Authors' own elaboration)

In the case of part (a), it is a true statement. The reasoning to establish its veracity is based on the definition of multiple of 7 and the distributive property of the product over the sum in the integers. Indeed, if a is a multiple of 7, then there is an integer n

such that $a = 7n$. Similarly, there is an integer m with $b = 7m$. In this way, the sum can be written as $a + b = 7n + 7m = 7(m + n)$, as $m + n$ is an integer number, then $a + b$ is a multiple of 7. We clarify that we did not expect such sophisticated reasoning as the one we have just presented, but we did expect some general arguments.

For (b), again, this is a true statement. To establish it, students needed to know how an odd number is generally expressed and that the product of two consecutive integers is an even number and some knowledge of algebra such as the development of a binomial squared and factorization. More precisely, let n be an odd number, then there is an integer k such that $n = 2k + 1$, then $n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + k = 4k(k + 1)$. From the latter expression, $k(k + 1)$ is an even number because it is the product of two consecutive integers, so that $4k(k + 1)$ is a multiple of 8, that is, $n^2 - 1$ is a multiple of 8. Here, the reasoning is more sophisticated than in the previous statement.

Part (c) is a false statement, a counterexample is sufficient to refute it. Although it is true for a "large number of cases" (for the first 10 natural numbers the expression gives a prime number), for the case of $n = 11$ is obtained 121, which is not a prime number, since precisely 11 divides 121.

The second activity was stated as follows (see **Figure 2**).

Activity 2: What is the last digit of 2^{50}?

Figure 2. Activity 2 of the diagnostic questionnaire (Source: Authors' own elaboration)

The purpose of this activity is to see if students can undertake an inductive procedure to discover a rule of behavior of powers of 2 and, from there, determine the solution of the exercise. The aim is for the student to observe that the last digits of the powers of 2 form a periodic sequence of size 4, formed by the numbers 2, 4, 8, and 6. From calculating small powers of 2. For example, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$, $2^6 = 64$, $2^7 = 128$, $2^8 = 256$. In this case the general rule can be constructed analytically as follows: Let $c(n)$ be the last digit of 2^n, then $c(n) = 2$ provided $n = 4k + 1$, $(k = 0, 1, 2, \dots)$; $c(n) = 4$ if $n = 4k + 2$, $(k = 0, 1, 2, \dots)$; $c(n) = 8$ if $n = 4k + 3$, $(k = 0, 1, 2, \dots)$ and $c(n) = 6$ whenever $n = 4k$, $(k = 0, 1, 2, \dots)$ or in simple terms, $c(n) = 2, 4, 8, 6$ if n dividing by 4, the respective remainders are equal to 1,2,3 and 0.

In the third activity, the following situation arises (see **Figure 3**).

Activity 3

Wonderland

Any similarity to real events (present, future or past) is a coincidence. This is a fiction.

After a "scientific" study it has been established that the following statements are true:

- If the dog is happy and there is a child nearby THEN it approaches the child and circles around it.
- If the rabbit sees the dog THEN it runs to take refuge in its den.
- If the dog approaches a person who is afraid THEN barks.
- If the dog bites THEN the dog is locked up for a week.
- If the sky is clear THEN the mouse walks under the tree.
- If the sky is clear THEN the rabbit tries to steal a carrot from the garden.
- If the rabbit tries to steal a carrot from the garden THEN the dog runs after it.
- If the dog is hit THEN he bites.

In a garden - in wonderland - there is a rabbit, a dog and a mouse walking under a tree. That morning, Alicia, a 6-year-old girl, is in the garden. She is afraid of animals and hits them when they are around her and make noise.

What is going to happen then in the garden?

Figure 3. Activity 3 of the diagnostic questionnaire (Source: Authors' own elaboration)

The objective of this activity was to find out if students are able to chain statements in a deductive way starting from certain premises and taking into account a set of statements that are considered true, regardless of whether they are mathematical content or not. In this case, the eight statements listed at the beginning are considered true and as premises we have that: It is morning, a rabbit, a mouse and Alice meet in the garden and Alice, out of fear, hits the animals if they are near her.

A possible deduction could be the dog approaches Alice and circles around her, as Alice is afraid, the dog barks at her and Alice hits him, so the dog bites her and is locked up for a week. However, different chains of statements can be made from the given statements.

Diagnostic questionnaire sheets were collected from the groups reported above for further analysis.

RESULTS

This section concentrates on the information about the answers proposed by the students, which are presented in tables.

Activity 1

State whether the following statements are true or false. Give as rigorous a justification as possible for your answers.

a. The sum of two multiples of 7 is a multiple of 7.

b. If n an odd number, then $n^2 - 1$ is multiple of 8.

c. For each integer number, the integer $n^2 - n + 11$ is a prime number.

Table 1 shows the results of activity 1 of the diagnostic questionnaire.

Table 1. Summary of responses to activity 1

		(a)	(b)	(c)
Correct answers	WJ	0% (0)	0% (0)	0 % (0)
	IJ	79% (35)	45% (20)	28 % (12)
	NJ	5% (2)	5% (2)	2 % (1)
Incorrect answers	J	7% (3)	41% (18)	57 % (25)
	NJ	0% (0)	2% (1)	4 % (2)
Another response		7 % (3)	7% (3)	5% (2)
He/She did not respond		2 % (1)	0% (0)	4% (2)
Total number of correct answers		84 % (37)	50% (22)	30% (13)
Total number of incorrect answers		7 % (3)	43% (19)	61% (27)

Details for reading Table 1

1. The number in parentheses in each cell is the total number of students who responded to the specific statement.
2. Students' responses were classified by first considering whether they were correct or incorrect, and then by considering whether their response was well justified (WJ, referring to justifications that draw on general reasoning), incorrectly justified (IJ, referring to justifications that draw on one or more particular examples) or not justified (NJ) (in the case of correct responses) or justified (J) or not justified (NJ) (in the case of incorrect responses).
3. The category of "other responses" includes those in which the students indicated that they had no knowledge, did not know what the mathematical content of the statements referred to, did not detail their response, although they carried out operations or wrote algebraic expressions, or performed incomplete calculations.
4. It is important to clarify what we mean by a "correct and well-justified answer". For the statement in (a), the meaning of "well justified" implies mentioning a definition of multiple of 7 and the distributive property of the product with respect to addition in the integers. The idea for the other parts is similar.

Table 1 shows that thirty-seven students (84%) considered the statement in (a) to be true, while three (7%) considered it to be false, another three (7%) provided "other answers" and one student (2%) did not respond. Thirty of the justifications provided by the students supporting the veracity of the statement were because the property is fulfilled for one or more particular cases without resorting to arguments of a general nature, an example of the students' justification is as shown in **Figure 4**.

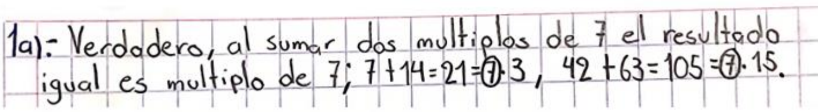


Figure 4. Example of a justification for the answer TRUE to the statement in (a) (Source: Authors' own elaboration)

Draws attention to the fact that this student presented an example with "large numbers" multiples of 7 (42 and 63) as if that fact gave more strength to his argument. On the other hand, five justifications provided by the students exhibited a diversity of reasons, addressing concepts such as multiplication and division as inverse operations, while two of the students limited their responses to the presentation of algebraic expressions. Two students who responded that the statement is true did not provide an explicit justification.

Another example of a justification for a TRUE answer to the statement in (a), is presented in **Figure 5**.

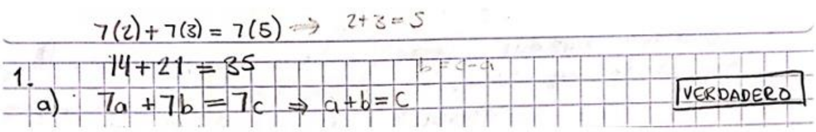


Figure 5. Another example of a justification for a TRUE answer to the statement in (a) (Source: Authors' own elaboration)

Here the student tried to make a general reasoning: she wrote $7a + 7b = 7c$, in the left-hand side of the equation took two arbitrary multiples of 7 (although she did not specify that a y b must be integer numbers) and on the right side she wrote $7c$ explaining that the result of the sum should also be a multiple of 7. It was observed that, although he tried to make a general reasoning, he does not conclude it and chooses to give a and b particular values.

Regarding the students who responded that the statement is false, one student presented the following justification:

Example of a justification for the FALSE answer to the statement in (a).

> “$1 + 7 = 8$, is not a multiple of 7.”

He chose the numbers 1 and 7 of which the first one is not a multiple of 7, calculates the sum and gets 8, which is not a multiple of 7. His strategy was based on providing a specific case in which the result does not meet the condition of being a multiple of 7, which he considered sufficient to support his answer. It seems that, this student confused the term multiple with the term divisor. On the other hand, another student used the numbers 4 and 3 as an example, arguing that the sum of these is 7 to satisfy the statement. This case reveals, first of all, a lack of understanding on the part of the student regarding the statement or potential confusion about what is implied by a number being a multiple of 7. However, contrary to what one might think in the sense that he is resorting to the “law of counterexample” to determine the falsity of the statement, it is not clear whether he is doing so because in thinking that a finite number of cases is sufficient to affirm the truth of a statement, the same must be true since a finite number of cases that do not verify a statement is sufficient to refute it. In addition, a single student stated that not all multiples complied with the statement, without providing additional details.

Students who gave “ANOTHER ANSWER”, expressed responses such as:

> “I am not very clear about what a multiple is.”

In the case of statement (b), according to the data in **Table 1**, twenty-two students (50%) responded that the statement is true. Nineteen (43%) considered it to be false. While three (7%) gave “other answers”. Twenty students of those who answered that the statement is true justified with particular examples as shown in **Figure 6**.

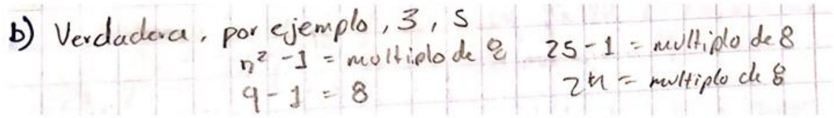


Figure 6. Example of a justification of a TRUE answer to the statement in (b) (Source: Field study)

The student claims that substituting odd numbers for the values yields numbers divisible by 8. Some students merely presented a single case, while others felt the need to present several cases as if that would provide a more convincing justification.

Of the nineteen students who considered the statement in (b) to be false, eighteen of them offered justifications, while one chose not to. Fifteen students supported their answers with numerical examples. That is to say, they substituted the number 1 in the expression $n^2 - 1$, others replaced the 4 and 8 (which are not odd). In this instance it is observed that they are not clear about the meaning of an odd number. Once they did the respective calculations they observed that, by replacing these certain numbers for this expression, the result obtained did not meet the condition of being a multiple of 8. These cases convinced students of the falsity of the claim.

The justification exhibited in **Figure 7** for the falsity of the statement is interesting.

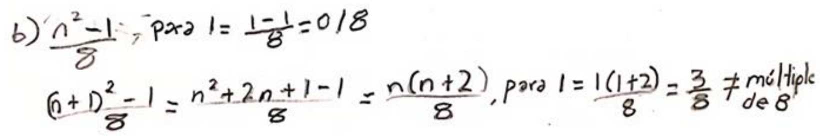


Figure 7. Example de justification provided by a student (Source: Field study)

There are several things that draw attention in the above justification. Although the student does not explicitly describe what a multiple is, there is an evident intention to explore the nature of the quotient $\frac{n^2-1}{8}$, which, if it is an integer, would imply that the numerator is a multiple of 8. Furthermore, the student checks whether this is true for $n = 1$ and then substitutes n in the initial formula with $n + 1$, as if attempting to demonstrate by mathematical induction. This situation is intriguing because, as previously mentioned, HSE in Mexico typically does not cover proof processes. However, in a brief interview with the student, he stated that he had read about mathematical induction proofs on his own and attempted to apply it to the example. When transforming the expression obtained by substituting n by $n + 1$, obtains a new expression $\frac{n(n+2)}{8}$ and then proceed by trial and error by giving n the value 1, obtaining $\frac{3}{8}$ and concluding that the claim is false. He does not realize that one of the conditions for the original expression to give a multiple of 8 is that n must be odd and when substituting by $n + 1$, an even number is obtained. The expression that obtains $\frac{n(n+2)}{8}$, is an integer provided n is an even number.

Students who provided “ANOTHER ANSWER”, expressed:

> “I do not know what a multiple of 8.”

Regarding the statement in item (c), it is evident in **Table 1** that twenty-seven students (61%) affirmed that the statement is true, while thirteen (30%) considered it false. In addition, two students (5%) provided “other answers”, and two others (4%) did not respond. Of the students who stated that item (c) is true, twenty-five offered justifications to support their answers, while two did not. Interestingly, twenty-four of the justifications provided followed a similar pattern to those in the previous parts.

Specifically, students substituted particular numbers (one or more, although they never reached 11) and, obtaining a prime number in all cases, concluded that the statement was true from justifications such as in **Figure 8**.

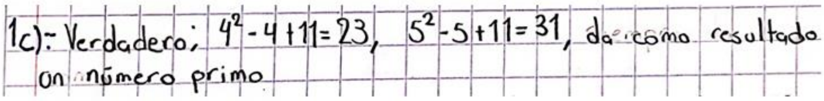


Figure 8. Example of a justification of a TRUE answer to the statement in (c) (Source: Field study)

On the other hand, the other four justifications were based on arguments such as "the result is odd, all numbers, all natural numbers are prime because they can be identified by name", without providing additional information in this regard.

Only thirteen of the students responded that the statement was false, of which eleven of them gave a justification and two did not. Of the eleven students, six argued the following:

Example of a justification of a FALSE answer to the statement in (c).

> "Because there is no formula for prime numbers."

When asked verbally why they made this claim, they replied that they had read that there were no formulas for obtaining prime numbers. A student made mistakes in their calculations, claiming that 5^2 plus 11 is not a prime number because 5^2 is 25, and $25 + 11$ is 35. Two students claimed that the sum of an integer and an odd number cannot be a prime number and another student justified his answer by using unfinished operations. One of the students claimed that the numbers being even makes them not prime. None of the students who answered that this statement is false resorted to a counterexample in their justification.

In **Figure 9**, it is shown that two students provided "OTHER ANSWERS" for the part (c).

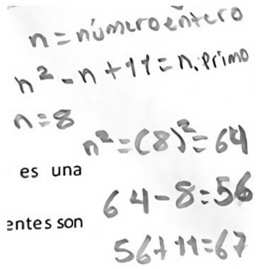


Figure 9. Example of "other answers" provided for two students (Source: Authors' own elaboration)

The student did not provide additional information. The other student wrote the expression $n^2 - n + 11 = i$, and transformed it into $n^2 - n + 11 = i$, then makes $n = 1$, and obtains $1 - 1 + 11 = i$, $i = 11$, but does not add any further comments.

Activity 2

What is the latest digit of 2^{50}?

The results of activity 2 are presented in **Table 2**, which can be read in a similar way as **Table 1**.

Table 2. Summary of responses to activity 2

	Correct answers			Incorrect answers		I don't know	He/she did not respond
	WJ	IJ	NJ	J	NJ		
Total	0 (0%)	14 (31%)	17 (39%)	1 (2%)	8 (18%)	2 (5%)	2 (5%)

Table 2 shows that thirty-one students (70%) responded correctly to the question in activity 2. On the other hand, nine students (20%) gave incorrect answers, while two (5%) indicated that they did not know, and another two students (5%) chose not to answer.

Of the students who gave 4 (correct answer), thirteen of them justified their answer, mainly through examples where they squared the first ten numbers, identifying a regularity between the last digits and, from this observation, they offered their response. Below (**Figure 10**) is one student's justification for why 4 is the last digit.

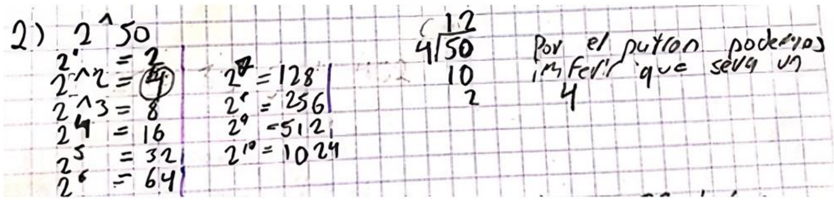


Figure 10. Example of a student's justification of his answer "4" in activity 2 (Source: Field study)

One student justified his answer by arguing:

> "Because what is being raised is divisible by 2."

The other seventeen students gave no justifications.

Regarding the incorrect answers, eight of them lacked justification. For example, two students provided the answer “3200” without offering any explanation. One stated that the correct answer was “15”, while another student wrote “2×25” equals “50”. Another student answered “8”, while another said “$2 \times 10^5 = 0$”. In this case, a lack of knowledge about operations is observed, particularly when working with powers. This is evident because the student expressed that the product of these numbers is zero. Another student answered with the expression 15^2. The remaining student wrote “2” as an answer without adding any explanation.

The student who did justify their answer expressed it in the following way:

“ $2^{50} = 98$

$2^{50} - 2 = 100 - 2 = 98.$”

In the justification given above, it is observed that the student confuses raising a number to a power with the product of 2 and 50. However, there is no evidence why he subtracts 2 from the result.

One of the students who claimed not to know the answer, performed these calculations (see **Figure 11**).

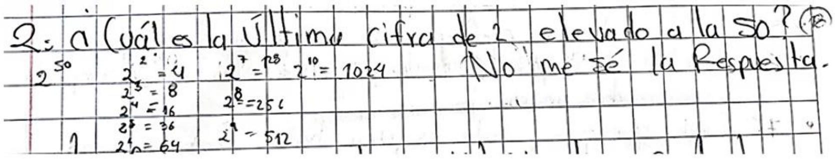


Figure 11. Example of justification in activity 2 (Source: Field study)

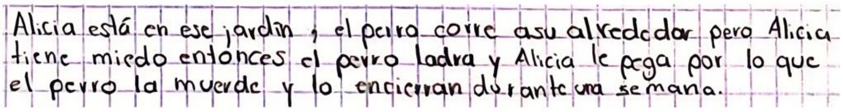
Regarding the justification given by this student, it is evident that they attempted to identify a pattern in the successive powers of 2. However, he fails to conclude.

Activity 3

Wonderland

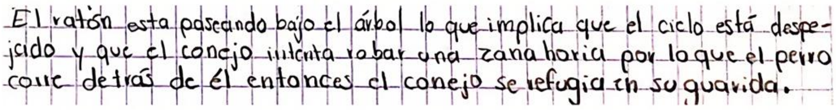
Of the forty-four students, two did not respond to the activity and three answered that nothing would happen, since it was a fictitious situation.

The remaining thirty-nine students constructed or attempted to construct deductive chains (see **Figure 12** and **Figure 13**), of which we highlight the following:



Alicia está en ese jardín y el perro corre a su alrededor pero Alicia tiene miedo entonces el perro ladra y Alicia le pega por lo que el perro la muerde y lo encierran durante una semana.

Figure 12. Example of a deductive chain provided in activity 3 (Source: Field study)

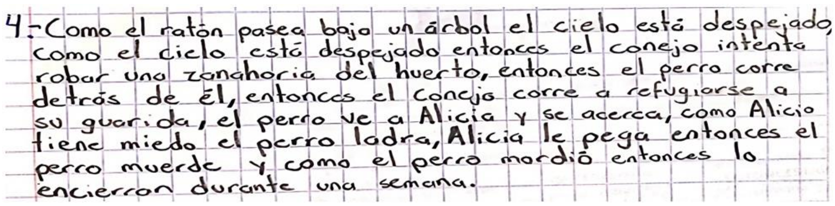


El ratón esta paseando bajo el árbol lo que implica que el cielo está despejado y que el conejo intenta robar una zanahoria por lo que el perro corre detrás de él entonces el conejo se refugia en su guarida.

Figure 13. Example of another deductive chain proposed in activity 3 (Source: Field study)

These two deductive chains were presented by five students. In the case of the first one, the chaining appears correctly. However, we observe that, in the second one, the student takes as a premise that the mouse is walking under the tree and from there infers that the sky is clear. But within the set of statements that are considered true (given at the beginning of the activity) is the one that says that, *if the sky is clear*, *THEN the mouse is walking under the tree*. That is, the student of this deduction is confusing an implication with its reciprocal; this situation was exposed by seven other students. This event occurs frequently in the teaching of mathematics and, in many cases, is due to the frequent error of interpreting a proposition as an equivalence or confusing it with its reciprocal.

These two types of deductive chains were presented (with slight variations) by several students. More precisely, two students presented only the first chain, another, the second, and two presented both, as shown in **Figure 14**.



4- Como el ratón pasea bajo un árbol el cielo está despejado, como el cielo está despejado entonces el conejo intenta robar una zanahoria del huerto, entonces el perro corre detrás de él, entonces el conejo corre a refugiarse a su guarida, el perro ve a Alicia y se acerca, como Alicia tiene miedo el perro ladra, Alicia le pega entonces el perro muerde y como el perro mordió entonces lo encierron durante una semana.

Figure 14. Example of a deductive chain with slight variations displayed in activity 3 (Source: Field study)

Some students presented their deductions in a more schematic, or abbreviated way without making the implications explicit. **Figure 15** is an example of such a situation.

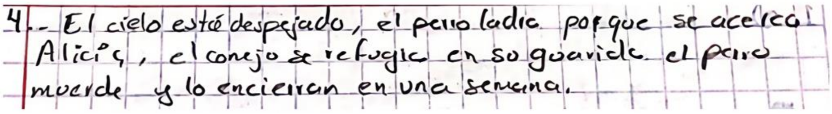

4.- El cielo está despejado, el perro ladra porque se acerca Alicia, el conejo se refugia en su guarida el perro muerde y lo encierran en una semana.

Figure 15. Example of a schematic deductive chain presented in activity 3 (Source: Field study)

Eleven of the students showed deductions similar to those in the previous example, which lacked more detailed explanations. This type of inference has different limitations, such as, for example, the failure to provide necessary details, preventing the reasoning from being adequately understood. By not expressing the premises explicitly, there is a risk of omitting details and incurring fallacies by using abbreviated assumptions without adequate care. At the university, particularly in the bachelor's degree in mathematics, this type of reasoning is considered lacking in mathematical rigor, which is essential for conducting proofs.

Other examples of deductive chains presented are the following:

> "All the facts of the statements would occur."

> "Nothing will happen because the dog, the rabbit and the mouse cannot coincide because the dog would chase them."

Ten students expressed similar responses to the previous ones, reflecting a variety of interpretations of the situation. Some participants accepted that everything related to the affirmations would happen, realistic attitudes based on the observation of animal nature, appreciation of violence, imagination, repercussions, consequences or reactions that students may have on a specific scenario.

Six students presented more extensive deductions (see **Figure 16**).

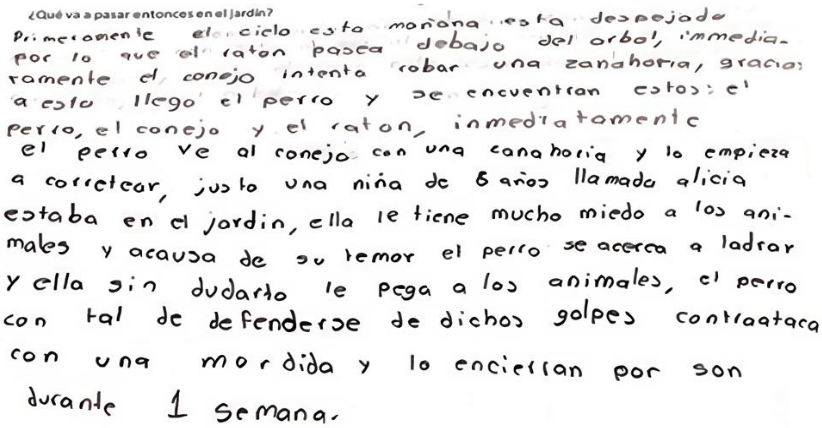

¿Qué va a pasar entonces en el jardín?
Primeramente el cielo esta mañana esta despejado por lo que el ratón pasea debajo del árbol, inmediatamente el conejo intenta robar una zanahoria, gracias a esto llegó el perro y se encuentran estos: el perro, el conejo y el ratón, inmediatamente el perro ve al conejo con una zanahoria y lo empieza a corretear, justo una niña de 6 años llamada alicia estaba en el jardín, ella le tiene mucho miedo a los animales y a causa de su temor el perro se acerca a ladrar y ella sin dudarlo le pega a los animales, el perro con tal de defenderse de dichos golpes contraataca con una mordida y lo encierran por son durante 1 semana.

Figure 16. Example of an extensive deductive chain provided in activity 3 (Source: Field study)

Deductions, like the above example, are crucial for clarity and understanding of the reasoning. Although the situation presented does not involve mathematical content, presenting or expressing deductions in an explicit way contributes to the construction of more convincing arguments by the students, to understand how to reach conclusions, develop reasoning and gain a deeper understanding of the subject matter.

One of the most extensive deductions is presented in **Figure 17**.

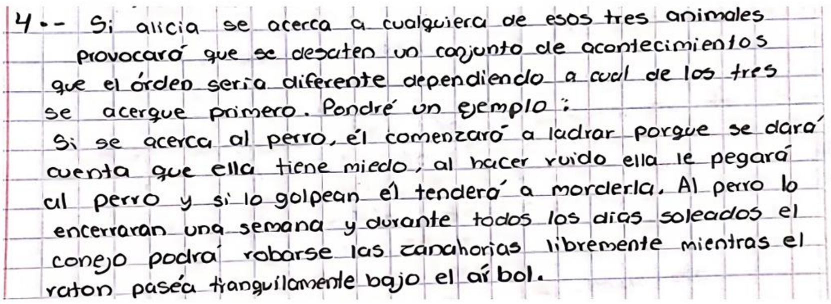

4.- Si alicia se acerca a cualquiera de esos tres animales provocará que se desaten un conjunto de acontecimientos que el órden sería diferente dependiendo a cual de los tres se acergue primero. Pondré un ejemplo:
Si se acerca al perro, él comenzará a ladrar porque se dará cuenta que ella tiene miedo, al hacer ruido ella le pegará al perro y si lo golpean él tenderá a morderla. Al perro lo encerraran una semana y durante todos los días soleados el conejo podrá robarse las zanahorias libremente mientras el raton paséa tranquilamente bajo el árbol.

Figure 17. Another example of an extensive deductive chain presented in activity 3 (Source: Field study)

In this example, the student addresses the situation and speculates on one of the possible cases that could occur. From his point of view there are others that can be developed depending on which of the characters is approached first. We can also observe that he understands the situation, identifies, and relates the provided data, formulates hypotheses or a possible answer, analyses individual cases and expresses them.

DISCUSSION AND CONCLUSION

After analyzing each of the answers given by the students to the activities that made up the questionnaire, we present here the following discussion: In relation to activity 1, 79% of the population, composed of 44 students, gave a correct answer but an incorrect justification in activity 1(a). For example, **Figure 4** and **Figure 5** show two productions that are considered correct and yet the students who produced them incorrectly justified their answers.

Although it is true that the objective was not to identify sophisticated reasoning, it was interesting to identify the type of reasoning that students put into play to solve the exercise. As can be identified in **Figure 4**, the student who produced such an answer only justified it with particular cases. Therefore, it is concluded that this percentage of students is capable of performing inductive procedures, although they do not state in an analytical way the rules found from the detection of regularities or patterns of behavior. One of the reasons may be due to the fact that commonly in pre-university, at least in Mexico, the teaching of mathematics at this level does not have mathematical proofs as an objective; generally, much weight is given to algorithmic work and numerical calculation, testing the validity of formulas or relations for particular cases, or taking formulas or mathematical properties for granted. On the other hand, in the productions, it was also identified that the level of argumentation is very limited; they are referred to support their answer in the fact that it is fulfilled for particular cases.

In activity 1(b), it was identified that 45% of the study population presented correct but incorrectly justified answers, as shown in **Figure 6**.

After the analysis of the answers considered as correct, but incorrectly justified, it can be inferred that this group of students does not have developed deductive reasoning skills, they proceed with particular cases and once the property is fulfilled for these cases, they take the generalization for granted. The level of argumentation they make for the justification is based on an algorithmic description, which corresponds to what has been done, and not being able to carry out a correct justification, perhaps due to the fact that they cannot express in a general way an odd number, basic operations with consecutive numbers and some notions of algebra such as developments of binomials squared and factorization. Therefore, this finding of difficulties shows the need to pay attention to the student's mathematical training, the need to develop argumentation and mathematical reasoning skills in the transition from pre-university to university level.

Figure 8 shows one of the productions of activity 1(c). Twenty-eight percent of the students produced this type of answers, as can be identified it is a verification for particular cases. Students were not able to identify that the proposition is false. A counterexample to the situation is presented when $n = 11$. In this case, the composite number 121 is obtained. However, when proceeding for particular cases the students assumed that the proposition is true. These findings reveal the absence of deductive reasoning skills.

After analyzing each of the productions that the students gave in response to activity 2, it was identified that 31% of the student population gave correct answers for particular cases; however, this number of participants did not correctly justify these answers. Meanwhile, 39% of the student population gave correct answers, but did not justify. Some of the students who justified incorrectly took as argument from developing some powers of the number 2 that the number sought is the number 4, as seen in **Figure 10**.

According to what was found as responses to activity 2, we can conclude that this group of students can use inductive procedures, although they do not articulate in an analytical way the rules identified from the detection of regularities or patterns of behavior of the powers of the number 2. In this case, students were expected to identify that the last digits of the powers of 2 generate a periodic sequence of size 4. It is also important to point out that in this group of participants who provided answers, the experience of working with situations that require the search for a pattern of behavior is absent; it is possible that they do not know the fundamental theorem of arithmetic, the development of powers, some basic divisibility criteria, among others. It was identified that these students have not had the experience of working with situations that require generalization, argumentation and proof.

Activity 3 produced several responses; in this case, it was not possible to organize them in a table due to the nature of the students' statements. It should be noted that the discussion on the answers to this activity are based on the eight premises established in the formulation of the activity, and in the formulation of the statement accepted as a valid answer by the authors of this research, which was formulated in the following terms:

> "the dog approaches Alice and circles around her, as Alice is afraid, the dog barks at her and Alice hits it, so the dog bites her and is locked up for a week."

In this direction, the answers shown in **Figure 12** and **Figure 13** were considered as correct. However, in spite of accepting these answers as correct, it cannot be omitted to point out that the level of deductive reasoning and argumentation that accompanies each of the productions in general is very low. Since it was possible to identify that the justification of the consistency of the answer to each statement is referred to a description, despite the fact that rigorousness in the justification is not required, the existence of difficulties in understanding the implication of each premise and the different ways of establishing the logical-mathematical relationships was identified.

In the answer shown in **Figure 12**, it can be identified that one of the premises considered by the main activity was formulated making the reciprocal formulation valid, which shows that the group of five students who formulated this approach as an answer does not rigorously consider the role played by the premises in the elaboration of deductive chains.

Two students made the statement identified in **Figure 14**. In the production it can be identified that this pair of students sought to establish more deductive chains in relation to the previous statements, this is important, because although it is inferred that these students in their passage through the pre-university did not promote in them the work with the mathematical proof and analysis of situations that involve argumentation and the implementation of levels of reasoning, in this case, deductive; they managed to establish the relationship of more premises in their statement.

Product of the analysis of the different answers that emerged, it can be established that during the explanations of the students in relation to their answers; it is identified that their level of deductive reasoning is low, for example, they do not explain in a solid way the role that the premises play, in the writing of the description of the answers, they usually consider the validity of the formulation of the reciprocal of some statements, they do not express in an explicit way some premises. These details can influence to incur in fallacies, these problems deserve an important attention, since, in higher education, particularly in mathematics, these aspects translate into a lack of mathematical rigor, which is essential to perform mathematical proof.

The answer to the question posed in activity 3 was limited. Even the absence of good deductive reasoning influenced the students to conclude that something will not happen in the garden, since everything is fictitious. The fictitious is true; however, as a response from the group of students who made up the study population, the aim was to find out how capable they were to establish statements by deductive chaining, which to a great extent was limited. This discussion was oriented more towards the correct and incorrectly justified answers, since the cases of correct and unjustified answers, or incorrect answers; from the point of view of the authors of this work definitely helped to strengthen the justification of the study of mathematical reasoning and argumentation, but it is in the study of the answers and the forms of justification where it was possible to find the various problems related to argumentation and reasoning that account for the relevance of the research.

Based on the findings about the difficulties in the development and use of inductive and deductive reasoning and about the low level of argumentation in general found in students recently admitted to university, the present work serves in part as a basis for a much broader research that is under development and that refers to the elaboration and implementation of a didactic proposal for the acquisition of argumentation and mathematical reasoning skills in students at university level, in addition to its exemplification in the study of some specific mathematical content. The projection under development requires that the categories of analysis that serve, as indicators of the development process of argumentation and mathematical reasoning skills be made explicit. Therefore, a broad discussion about the theoretical model of mathematical reasoning proposed by Jeannotte (2015) and Jeannotte and Kieran (2017) is in process, with the purpose of establishing the theoretical bases and justifying its use in the elaboration of the research under development.

The various problems identified in the students' responses form a field suitable for study from the perspective of mathematics education. In this direction, important research can be oriented and developed to influence the development of skills on these topics. It should be noted that these types of skills have a direct influence on the understanding of mathematical content, either by the teacher or by the student. Therefore, an innovative strategy could be proposed by considering the elements of Jeannette's model as the main indicators and based on them, to propose activities for the teaching and learning of mathematics

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Declaration of interest: No conflict of interest is declared by the authors.

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REFERENCES

Arsac, G. (1996). Un cadre d'étude du raisonnement mathématique [A mathematical reasoning framework]. In D. Grenier (Ed.), *Séminaire didactique et technologies cognitives en mathématiques* [*Seminar on didactics and cognitive technologies in mathematics*]. IMAG.

Arsac, G., & Mante, M. (1997). Situations d'initiation au raisonnement déductif [Situations to introduce deductive reasoning]. *Educational Studies in Mathematics, 33*(1), 21-43. https://doi.org/10.1023/A:1002967421661

Balacheff, N. (1987). Processus de preuves et situations de validation [Proving processes and situations for validation]. *Educational Studies in Mathematics, 18*(2), 147-176. https://doi.org/10.1007/BF00314724

Benítez, A. A., Benítez, H., & García, M. L. (2016). La argumentación sustancial. Una experiencia con estudiantes de Nivel Medio Superior en clases de matemáticas [Substantial argumentation. An experience with high school level students in math classes]. *Educación Matemática* [*Mathematics Education*], *28*(3), 175-216. https://doi.org/10.24844/em2803.07

Bravo, M., Martínez, J., & Arteaga, E. (2001). El valor formativo de las demostraciones [The educational value of proofs]. *Xixím: Revista Electrónica de Didáctica de las Matemáticas* [*Xixím: Electronic journal of didactics of mathematics*], *3*, 20-23.

Brodie, K. (2010). Mathematical reasoning through tasks: Learners' responses. In K. Brodie (Ed.), *Teaching mathematical reasoning in secondary school classrooms* (pp. 43-56). Springer. https://doi.org/10.1007/978-0-387-09742-8_3

Brousseau, G., & Gibel, P. (2005). Didactical handling of students' reasoning processes in problems solving situations. In C. Laborde, M. J. Perrín-Glorian, & A. Sierpinska (Eds.), *Beyond the apparent banality of the mathematics classroom* (pp. 13-58). Springer. https://doi.org/10.1007/0-387-30451-7_2

Castro, E., Cañadas, M. C., & Molina, M. (2010). El razonamiento inductivo como generador de conocimiento matemático [Inductive reasoning as a generator of mathematical knowledge]. *UNO: Revista de la Didáctica de la Matemática* [*UNO: Journal of Mathematics Education*], *54*, 55-67.

Codina, A., & Lupiáñez, J. (1999). *El razonamiento matemático: Argumentación y demostración* [*Mathematical reasoning: Argumentation and proof*]. http://cumbia.ath.cx:591/pna/Archivos/CodinaA99-2672.PDF

Conner, A. M., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics, 86*(3), 401-429. https://doi.org/10.1007/s10649-014-9532-8

Crespo, C. (2005). La importancia de la argumentación matemática en el aula [The importance of mathematical argumentation in the classroom]. *Premisa* [*Premise*], *24*, 23-29.

Di Martino, P., & Gregorio, F. (2019). The mathematical crisis in secondary-tertiary transition. *International Journal of Science and Mathematics Education, 17*, 825-843. https://doi.org/10.1007/s10763-018-9894-y

Durand-Guerrier, V., Boero, P., Douek, N., Epp, S. S., & Tanguay, D. (2012). Examining the role of logic in teaching proof. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in mathematics education. The 19th ICMI study* (pp. 369-389). Springer. https://doi.org/10.1007/978-94-007-2129-6_16

Duval, R. (1992). Argumenter, demontrer, expliquer: Continuité ou rupture cognitive? [Arguing, proof, explaining: Cognitive continuity or rupture?] *Petit X, 31*, 37-61.

Fabert, C., & Grenier, D. (2011). Une étude didactique de quelques éléments de raisonnement mathématique et de logique [A didactic study of some elements of mathematical reasoning and logic]. *Petit X, 87*, 31-52.

Flores, Á. (2007). Esquemas de argumentación en profesores de matemáticas del bachillerato [Argumentation schemes in high school mathematics of high school mathematics]. *Educación Matemática* [*Mathematics Education*], *19*(1), 63-98. https://doi.org/10.24844/EM1901.03

Geisler, S., & Rolka, K. (2021). "That wasn't the math I wanted to do!"–Students' beliefs during the transition from school to university mathematics. *International Journal of Science and Mathematics Education, 19*(3), 599-618. https://doi.org/10.1007/s10763-020-10072-y

Godino, J. D., & Recio, Á. M. (2001). Significados institucionales de la demostración: implicaciones para la educación matemática [Institutional meanings of proof. Implications for mathematics education]. *Enseñanza de las Ciencias* [*Science Education*], *19*(3), 405-414. https://doi.org/10.5565/rev/ensciencias.3991

Gómez-Chacón, I. M. (2009). Actitudes matemáticas: propuestas para la transición del bachillerato a la universidad [Mathematical attitudes: proposals for the transition from high school to university]. *Educación Matemática* [*Mathematics Education*], *21*(3), 5-32.

Hadamard, J. (1945). *The psychology of invention in the mathematical field.* Princeton University Press.

Huidobro, J. A., Méndez, M. A., & Serrano, M. L. (2010). Del Bachillerato a la Universidad: las Matemáticas en las carreras de ciencias y tecnología [From high school to university: mathematics in science and technology careers]. *Aula Abierta* [*Open Classroom*], *38*(1), 71-80.

Jeannotte, D. (2015). *Le raisonnement mathématique: proposition d'un modèle conceptuel pour l'enseignement et l'apprentissage au primaire et au secondaire* [*Mathematical reasoning: a conceptual model for learning and teaching at the primary and secondary levels of schooling*]. [PhD thesis, Université du Québec à Montréal]. https://archipel.uqam.ca/8129/1/D2927.pdf

Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics*, *96*, 1-16. https://doi.org/10.1007/s10649-017-9761-8

Jiménez, M., & Areizaga, A. (2001). Reflexiones acerca de los obstáculos que aparecen, en la enseñanza de las matemáticas, al pasar del bachillerato a la universidad [Reflections on the obstacles that appear, in the teaching of mathematics, when passing from high school to university]. In *Acta de* IX *Jornadas de ASEPUMA* [Proceedings of the 9th ASEPUMA Conference] (pp. 1-10). https://www.researchgate.net/publication/26428246_Reflexiones_acerca_de_los_obstaculos_que_aparecen_en_la_ensenanza_de_las_matematicas_al_pasar_del_Bachillerato_a_la_Universidad

Lakatos, I. (1978). *Pruebas y refutaciones. La lógica del descubrimiento matemático* [*Proofs and refutations: The logical of mathematical discovery*]. Alianza.

Larios, V. (2003). Si no demuestro... ¿enseño Matemática? [If I do not prove... do I teach mathematics?]. *Educación Matemática* [*Mathematics Education*], *15*, 163-178. https://doi.org/10.24844/EM1502.07

Larios, V. (2019). Propuesta de herramientas metodológicas para el análisis de la enseñanza y el aprendizaje de la demostración en la escuela [Proposal of methodological tools for the analysis of the teaching and learning of proof in schools]. In J. E. Marmolejo, & G. Moreno (Eds.), *La* demostración *matemática en contexto escolar* [*Mathematical proof in a school context*] (pp. 13-41). Universidad Autónoma de Guerrero.

Larrazolo, N., Backhoff, E., & Tirado, F. (2013). Habilidades de razonamiento matemático de estudiantes de educación media superior en México [Mathematical reasoning skills of high school students in Mexico]. *Revista Mexicana de Investigación Educativa* [*Mexican Journal of* Educational *Research*], *18*(59), 1137-1163.

Liebendörfer, M., & Schukajlow, S. (2017). Interest development during the first year at university: do mathematical beliefs predict interest in mathematics? *ZDM Mathematics Education*, *49*, 355-366. https://doi.org/10.1007/s11858-016-0827-3

Marmolejo, J. E., & Moreno, G. (2019). Demostración en contexto escolar [Proof in school context]. In J. E. Marmolejo, y G. Moreno (Eds.), *La demostración matemática en contexto escolar* [*Mathematical proof in a school context*] (pp. 43-63). Universidad Autónoma de Guerrero.

Mata-Pereira, J., & da Ponte, J. P. (2017). Enhancing students' mathematical reasoning in the classroom: Teacher actions facilitating generalization and justification. *Educational Studies in Mathematics, 96*(2), 169-186. https://doi.org/10.1007/s10649-017-9773-4

NCTM. (2000). *Principles and standards for school mathematics*. NCTM.

Niswah, U., & Qohar, A. (2020). Mathematical reasoning in mathematics learning on pyramid volume concepts. *Malikussaleh Journal of Mathematics Learning, 3*(1), 23-26. https://doi.org/10.29103/mjml.v3i1.2400

OCDE. (2017). *Marco de Evaluación y de Análisis de PISA para el desarrollo: Lectura, Matemáticas y Ciencias* [*Framework for Assessment and Analysis of PISA for Development: Reading, Mathematics, and Science*]. OCDE. https://www.oecd.org/pisa/aboutpisa/ebook - PISA-D Framework_PRELIMINARY version_SPANISH.pdf

O'Daffer, P. G., & Thornquist, B. A. (1993). Critical thinking, mathematical reasoning and proof. In P. S. Wilson (Ed.), *Research ideas for the classroom: High school mathematics* (pp. 31-40). MacMillan Publishing Company.

Pedreros, A. (2017). *Desarrollo de habilidades: Aprender a pensar matemáticamente* [*Skill development: Learning to think mathematically*]. Ministerio de Educación de Chile. https://bibliotecadigital.mineduc.cl/handle/20.500.12365/2433

Polya, G. (1966). *Matemáticas y razonamiento plausible* [*Mathematics and plausible reasoning*]. Trillas.

Polya, G. (1967). *La découverte des mathématiques* [*The discovery of mathematics*]. Dunot.

Posso, A. E., Gómez, J. D., & Uzuriaga, V. L. (2007). Dificultades que aparecen en el proceso enseñanza-aprendizaje de la matemática al pasar del bachillerato a la universidad [*Difficulties that appear in the teaching-learning process of mathematics when passing from high school to university*]. *Scientia et Technica* [*Science and technology*], *2*(34), 495-499.

Rach, S., & Heinze, A. (2016). The transition from school to university in mathematics: Which influence do school-related variables have? *International Journal of Science and Mathematics Education*, *15*(7), 1343-1363. https://doi.org/10.1007/s10763-016-9744-8

Sánchez-Matamoros, G., Adamuz-Povedano, N., Cañadas, M. C., Fernández-Ahumada, E., García, M. T., Moreno, A., Ramírez-Uclés, R., & Serradó, A. (2022). Matemáticas en el bachillerato [*High school* mathematics]. In L. J. Blanco, N. Climent, M. T. González, A. Moreno, G. Sánchez-Matamoros, C. de Castro, & C. Jiménez (Eds.), *Aportaciones al desarrollo del currículo desde la investigación en educación matemática* [*Contributions to curriculum development from research in mathematics education*] (pp. 199-223). Editorial de Granada. https://www.researchgate.net/publication/364652381_Matematicas_en_el_Bachillerato

Saorín, A., Torregrosa, G., & Quesada, H. (2019). Razonamiento configural y organización discursiva en procesos de prueba en contexto geométrico [Configural reasoning and discursive organization in proof processes in geometrical context]. *Revista Latinoamericana de Investigación en Matemática Educativa* [*Latin American Journal of Educational Mathematics Research*], *22*(2), 213-244. https://doi.org/10.12802/relime.19.2224

SEP. (2011). *Programas de estudio 2011. Guía para el Maestro. Educación Básica. Secundaria. Matemáticas* [*Study Programs 2011. Teacher's Guide. Basic Education. Secondary Level. Mathematics*]. SEP. https://www.secundariagraef.mx/maestros/ebooks/2011Matematicas.pdf

SEP. (2017a). *Aprendizajes clave. Plan y programas de estudio para la educación básica* [*Key learning. Plan and programs of study for basic education*]. SEP. https://www.sep.gob.mx/work/models/sep1/Resource/10933/1/images/Aprendizajes_clave_para_la_educacion_integral.pdf

SEP. (2017b). *Planes de estudio de referencia del MCC de la Educación Media Superior* [*Reference curriculum plans of the MCC for upper secondary education*]. SEP. https://www.gob.mx/cms/uploads/attachment/file/241519/planes-estudio-sems.pdf

Soler-Álvarez, M., & Manrique, V. (2014). El proceso de descubrimiento en la clase de matemáticas: Los razonamientos abductivo, inductivo y deductivo [Discovery process in mathematics class: Abductive, inductive and deductive reasoning]. *Enseñanza de las Ciencias* [*Science Education*], *32*(2), 191-219. https://doi.org/10.5565/rev/ensciencias.1026

Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal, 20*(2), 5-24. https://doi.org/10.1007/BF03217474

Torregosa, G., Quesada, H., & Penalva, M. C. (2010). Razonamiento configural como coordinación de procesos de visualización [Configural reasoning as coordination of visualisation process]. *Enseñanza de las Ciencias* [*Science Education*], *28*(3), 327-340. https://doi.org/10.5565/rev/ec/v28n3.187