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Designing the *encounter* of teachers with meanings and levels of understanding of the derivative

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ABSTRACT

Previous research has shown that mathematics teachers face difficulties teaching the derivative of a function at a point. One of these difficulties revolves around the lack of materials and specialized information on this topic that allows teachers to carry out this task. Seeking to contribute to overcoming this difficulty, we describe the design of an activity oriented towards the encounter of mathematics teachers with meanings and levels of understanding of the derivative of a function at a point. The design is theoretically founded on the concepts of learning and activity of the theory of objectification and incorporates relevant information in the analytical, analytical-numerical, and geometric meanings of the derivative as well as the intra, inter, and trans levels of understanding of the concept. Based on these references, we describe the tasks that make up our design. We see our design as contributing to the field of mathematics education regarding the promotion of in-service teachers' learning from an historical-cultural educational perspective.

Keywords: professional development, task design, derivative, joint labor

INTRODUCTION

It is widely accepted that differential and integral calculus have contributed tremendously to the development of fields as diverse as engineering, economics, biology and even the social sciences. To a large extent, the fundamental laws and concepts of this branch of mathematics allow us to describe, measure and even predict changes in nature. For example, through the derivative it is possible to quantify, describe and predict the speed at which variations occur in concrete reality. Although historically the notion of the derivative was developed in close relation to the idea of variation of natural processes (Fuentealba et al., 2022; Gutiérrez Mendoza et al., 2017; Ímaz Jahnke & Moreno Armella, 2009), its study began in classical Greece in the midst of some fundamental mathematical problems, such as those surrounding tangents to curves and quadrature, which were solved without using general methods (Gavilán Izquierdo, 2005). It was not until the 18th century that Newton and Leibnitz laid the foundations for what is known as differential and integral calculus, making the derivative seen as a rate of change (Wenzelburger, 1993) in problems associated with distances, velocities, and optimization.

The historical relevance of the derivative has led to its incorporation in diverse curricular plans around the world, not only at the university level but also in prior schooling (Desfitri, 2016; Panero et al., 2016; Tapiero, 2020). In the case of Chile, this concept is taught for the first time to students in their third and fourth years of secondary education (age 16-age 17) through the limits, derivatives, and integrals (LDI) course (MINEDUC, 2021) of the differentiated curricular plan of some schools of humanist-scientific character. Although these curricular plans offer guidelines for teaching the derivative in the classroom, we cannot forget what scientific research indicates as recurring difficulties in the teaching of this concept.

Among these difficulties, various authors (González et al., 2013; González-García et al., 2018; Gutiérrez Mendoza et al., 2017; Hitt, 2003; Orhun, 2012) refer to a lack of attention to the derivative from a graphic perspective that keeps it disconnected from its symbolic expression, a purely algorithmic manipulation of this expression which leads to mechanized calculations, and a traditional teacher discourse characterized by transmissive teaching of the derivative. Even when the close relationship of these difficulties with the teaching of the derivative is well documented, it is notable that research on the learning of this concept continues to have its focus more on the figure of the student rather than on the teacher (see e.g., Saraza & Prada-Núñez, 2017) despite the fact that researchers such as Moreno (2005) have emphasized the importance of shifting this focus.

These difficulties are exacerbated with the curricular changes in recent years in some countries (Hitt, 2017). With regards to Chile, such changes have placed mathematics teachers in a complicated situation, given the conditions in which they undertake teaching the derivative in the schools where they work[1]. In general, teachers that impart classes that make up the general education curriculum in schools across the country count on the support of a variety of official curricular materials (school texts, activity notebooks, and teaching guides, among others). However, these conditions are not the same for general education curriculum courses related to humanist-scientific education. Regarding the LDI course, teachers only have general orientations provided by the course syllabus (MINEDUC, 2021) at their disposal, which increases the probabilities that these subjects orient their teaching of the derivative towards two dominant models: a more formal model, characterized by classroom work based on definitions, theorems and standard problems, or a more operational one, where the exercise of algorithms is privileged (Cuevas Vallejo & Pluvinage, 2009).

In the face of this reality, we see it necessary to promote training activities that support teachers charged with teaching the fundamental concepts of calculus, particularly the derivative, in overcoming the difficulties mentioned and contending with the conditions they face upon teaching this concept. From our perspective, this could benefit from the encounter of the teachers with *research results* about the learning of the derivative of a function at a point, as well as with other *voices* from which they can produce creative and subversive ways of teaching this concept in the classroom. Regarding the research results, we focus on those that link the learning of the derivative with specific meanings and levels of understanding of this concept.

Regarding the meanings of the derivative, the LDI course presents as a learning objective the modeling of phenomena that involve instantaneous rate of change and evaluation of the eventual necessity to adjust the obtained model (MINEDUC, 2021). From this, we consider it possible that the teacher recognizes in this objective a particular orientation of the derivative towards its most analytical aspect (García-Cuéllar & Martínez-Miraval, 2023; Gutiérrez Mendoza et al., 2017; Sánchez-Matamoros, 2014), putting aside other, equally important, meanings of the concept. Furthermore, upon lacking materials that allow them to interpret how students understand the concept of the derivative and devise pedagogical actions that evolve this understanding, teachers may find valuable the use of the levels of understanding of the concept that is found in specialized literature.

For these reasons, and from a historical-cultural educational approach, in this article we describe the design of a training activity oriented towards the encounter of mathematics teachers with certain meanings and levels of understanding of the derivative at a point.

THEORETICAL REFERENCES

The training activity that we propose is founded in the concepts of learning and activity of the theory of objectification (TO) (Radford, 2014, 2021) and incorporates theoretical information regarding the meanings and levels of understanding of the concept of the derivative at a point (Sánchez-Matamoros, 2014). Below, we present these theoretical references and employ them for the purposes of our design.

Learning and Activity

The TO conceptualizes *learning* of mathematics (or artistic, scientific, pedagogical, or other) content in terms of processes that consist of both knowing and becoming. According to Radford (2021), learning

> [...] is a continuous and tense encounter of mutual dialectical transformation between a cultural world (i.e., a cultural world that transcends the individual as a unique individual) and unique individuals who encounter it. In the course of this fusion, the world that appears to consciousness and the consciousness that arises from this encounter are continually transformed (p. 83).

Two ideas of interest for our design follow from the preceding quote. First, learning implies the *encounter* between the historical-cultural forms of thinking (knowledge) and the emerging consciousnesses that seeks to perceive them (beings). Second, learning implies the mutual *transformation* of the historical-cultural forms and the consciousnesses that encounter each other in this process: regarding the forms, these turn into objects of consciousness for individuals; regarding the consciousnesses, these are transformed into unique subjectivities that do not unconditionally adopt these forms of thinking, but rather position themselves critically before them (Radford 2021).

From these ideas, in this article, we are interested in certain forms of historical-cultural thinking regarding the derivative at a point. Specifically, we place the attention on some meanings and levels of understanding of the concept that are described in the following section. In the same way, we assume mathematics teachers who are involved in continuous education processes to be emerging consciousnesses that seek to perceive and give meaning to the aforementioned historical-cultural forms. From these considerations, in our design, we assume learning to be the progressive encounter of mathematics teachers with certain meanings and levels of understanding of the derivative at a point, in search of the transformation of these historical-cultural forms into an object of consciousness for them and of these same individuals into those unique subjectivities that position themselves critically before the knowledge they encounter.

This learning occurs within the limits and possibilities of the collective *activity* that is carried out by the teachers and the teacher educator (TE) during the formative process. In the TO, the activity is conceptualized not as a mere set of actions undertaken to accomplish an objective, but rather as a social and collective process through which individuals produce their own

[1] It is important to emphasize that this issue is not unique to Chile, as evidenced by the extensive literature in mathematics education concerning the teaching of derivatives (see e.g., Gavilán-Izquierdo & Gallego-Sánchez, 2021; Moreno-Armella, 2021). We cite the case of Chile here due to its relevance as the specific context of our study.

means of subsistence while also producing themselves as human beings (Radford, 2020). Radford (2021) defines the activity as *joint labor,* that is, "as a collective *form of life*, an *energy* that the participants release and the comes to envelop them, a special-temporal energy that is sensible and sensual, material and ideational, discursive and gestural, a flux carrying intentions, desires, and motive" (pp. 90-91).

This quote highlights two characteristic aspects of joint labor according to the TO. On the one hand, that this activity is a form of life means that it is a process in which subjects, when working shoulder to shoulder during their encounter with historical-cultural forms of things, experience self-expression, intellectual/social development, and aesthetic pleasure (Radford, 2017). On the other hand, joint labor makes possible the progressive encounter with historical-cultural knowledge that is not reduced to a cognitive or mental activity, but rather also addresses a flow of energy that is simultaneously embodied, discursive, subversive, affective, symbolic, and material (Radford, 2020).

The aforementioned suggests that a continuous training activity for mathematics teachers can become joint labor under certain conditions. In our design, we call activity or joint labor the flow of energy that comes from the teachers and the TE upon working together and responsibly committing to making certain meanings and levels of understanding of the derivative apparent in the classroom. Wrapped up in this flow of energy, the subject transforms these meanings and levels into objects that can be recognized by consciousness, at the same time that they are transformed into ethical and responsible subjects before the task of teaching the concept of the derivative in their schools. In this way, the meanings and levels of understanding of the derivative appear sensuously in the activity through perception, gestures, language and the use of signs and artifacts, giving rise to what Radford (2021) calls *common work*.

Meanings and Levels of Understanding of the Derivative

The derivative at a point is a mathematical concept that has developed historically to such an extent that it is possible to approach the concept from different *meanings* that have been attributed to it. In the field of mathematics education, researchers have dedicated themselves to understanding how the learning of this concept is produced based on its meanings and the relationships between them. For example, Sánchez-Matamoros and Fernández (2016) highlight three of these meanings: *analytical, analytical-numerical,* and *graphical* (henceforth, *geometric*). In this regard, the derivative of a function f at point $x = a$ is linked:

- To the *limit* of the average rate of change of f in a very small interval around $x = a$, as the size of the interval approaches zero (analytical meaning).
- To the *use* of analytical and numerical methods of approximating the instantaneous rate of change of f at $x = a$ (analytical-numerical meaning).
- To the *slope* of the tangent line of the curve of f at $x = a$ (geometric meaning).

In this article, we understand that these three meanings underlie not only the course syllabus itself, but also the problems that teachers select from the curricular materials to promote the learning of this concept. Based on the above, we assume that a training activity that seeks to bring teachers closer to the analytical, analytical-numerical, and geometric meanings of the derivative must consider and promote the establishment of relations of these meanings with certain problems involving the derivative that mobilize the meanings associated with the concept.

From these meanings, the treatment of the derivative at a point has allowed researchers to study the way in which learning of this concept occurs, in close relation to a series of *levels of understanding* of the derivative. In this article, we are interested in those levels of understanding denominated *intra, inter*, and *trans.* Originating in studies that deal with the development of *schemes*[2], these levels provide information related to characteristics in one's understanding of the derivative at a point. For Sánchez-Matamoros et al. (2012), *a student* finds himself or herself on level:

- Intra, when he or she uses elements of the derivative, associated with one or some of the meanings, and is not able to relate them when solving the problems.
- Inter, when he or she uses elements of the derivative, associated with one or some of the meanings, and is able to relate them when solving the problems.
- Trans, when he or she uses all elements of the derivative, associated with all three meanings, relating them when solving the problems.

From the description of these three levels, some important aspects are highlighted. First, the elements of the derivative refer to the *set of specific mathematics contents,* coming from the different meanings of the concept, that is necessary for responding to a specific problem about derivatives at a point (Sánchez-Matamoros, 2004, 2014). Second, the three levels can be identified in the *analysis of responses* of the students to the problems they are presented with. Finally, the students reach higher levels when they are able to *use more elements* of the derivative and *relate them* when they solve the problems.

In summary, we consider that a training activity oriented towards the encounter of teachers with the meanings and levels of understanding of the derivative must create conditions that allow the teachers not only to identify the intra, inter, and trans levels in students' responses to problems, but also to position themselves critically before the use of these levels when the focus is on the collective activity carried out by the students.

[2] Mathematical structures formed by mathematical elements and the relations that are established between them, which can be called upon for the solution of a given problem (Sánchez-Matamoros et al., 2008).

METHODOLOGICAL REFERENCES

The configuration of the activity we designed is structured according to the two interrelated components of every teaching-learning activity: Φ and Θ (Radford, 2021). Below, we describe these components for the purposes of our design.

Component Φ

Component Φ refers to the didactic organization of the activity, with attention to the *object-goal-task* structure. In Radford's (2021) words, "the teaching-learning activity has an *object*. […] For the activity to develop in the direction of its object, one or more *goals* can be identified. […] To achieve the goals of the activity, a specific *task* can be devised" (pp. 88-89).

The object confers a certain orientation to the activity, allowing it to take on a material or ideal form in concrete practice (Leontiev, 1984). In our case, the activity we designed has as its *object* the encounter of the mathematics teachers with certain historical-cultural forms of thinking about the derivative at a point. We refer to the analytical, analytical-numerical, and geometric meanings of the derivative as well as the intra, inter, and trans levels of understanding of this concept. For the training activity to be oriented towards this object, we describe the following *goals*:

1. Establish relations between problems of the derivative at a point and the analytical, analytical-numerical, and geometric meanings associated with this concept.
2. Establish relations between students' responses to problems concerning the derivative at a point and the intra, inter, and trans levels of understanding of this concept.
3. Establish a position regarding the use of the intra, inter, and trans levels of understanding to analyze interactions of students who together solve a problem concerning the derivative at a point.

Finally, the goals of the activity can be achieved by completing specific tasks (composed of increasingly complex problems and questions) that the students are invited to complete (Radford, 2015). In our case, we created three interrelated *tasks,* one for each goal of the activity. To do so, we leaned on some of the considerations for the choice of mathematical problems, the sequencing of these problems and the social organization of the classroom aimed at their resolution (Radford, 2021). The considerations that we considered when preparing the designed tasks were the following:

- *What the teachers know regarding the derivative and its teaching-learning*. The tasks account for the teachers' familiarity with the derivative at a point (that is to say, with the set of contents associated with this concept) at the moment of:
  - selecting problems in accordance with a learning objective,
  - scoring student responses to these problems, and
  - suggesting forms of didactic intervention that favor a better understanding of the concept.
- *The use of theoretical information about the meanings and levels of understanding of the derivative*. The tasks invite the use of theoretical information that comes from studies by Sánchez-Matamoros (2004, 2014) and her team regarding the learning of the concept of the derivative with students in their final year of secondary education in Spain. This information was already discussed in the section on theoretical references.
- *The teachers' interest in completing to the proposed tasks*. The tasks are framed in a professional context that can be interesting to mathematics teachers that teach the concept of the derivative to secondary school students in Chile. We incorporate this context in the tasks through a narrative that revolves around the practical experience of Simón (fictitious character), a mathematics teacher in charge of the LDI course for students in the third and fourth years of secondary education. The narrative shows Simón in different situations of planning the teaching and evaluation of the students' learning with respect to the derivative at a point. The purpose of this context is to ensure that the proposed problems and their questions make sense to the teachers who participate in the activity, driving them to solve the tasks.
- *The growing complexity of the tasks and problems*. Each task is phrased in such a way that the associated questions urge progressively deeper reflection from the teachers regarding the meaning and levels of understanding of the derivative, with attention to certain situations of teaching the concept that merit consideration.
- *The encounter with other voices and consciousnesses*. Given that forms of human collaboration are important pieces of every teaching-learning activity (Radford, 2021), we seek that the tasks provoke the encounter with the other between the teachers and the TE during the activity. In this sense, we imagine the solving of the tasks as non-alienating spaces for reflection, exchange, and debate, in which the teachers position themselves critically and act responsibly in the face of the opinions of others.

Component Θ

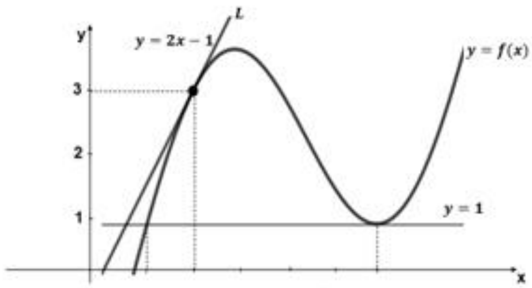
The component Θ refers to the activity or joint labor in and of itself, that is, to the process "that *materializes* knowledge into something intelligible" (Radford, 2021, p. 90). In our design, Θ is understood as every activity in which, as TEs, we hope that the didactic project as defined by the aforementioned component Φ is made a reality. Nevertheless, we recognize that the didactic organization in question does not take precedence over the way the activity will develop in concrete practice. In fact, the activity seen as a dynamic and complex process is subject to change during its unfolding, which makes it unpredictable from start to finish. Even so, we identify some *moments* through which the training activity passes, while it is oriented towards its object.

Based on the moments of the teaching-learning activity suggested by Radford (2021), we propose the following for our design:

(i) *presentation of the task,* where the task is shared with the teachers and any questions or concerns are addressed, Simón is a mathematics teacher that has taught his first five years at a humanist-scientific school. This year, he began teaching the *limits, derivatives, and integrals* course to students in their third and fourth years of secondary school. After a few classes on the *modeling of situations involving instantaneous rate of change and adjusting the obtained model* (LO3 of the course syllabus), Simón wishes to implement an assessment to monitor the progress of the students regarding the achievement of this objective.

While looking for possible problems in textbooks, Simón found the following:

Suppose that line L is tangent to the graph of the function at point $(2,3)$ as shown in the figure. Find $f(2)$ and $f'(2)$.



(a)

Verify that the average rate of change of $f(x) = \frac{x+3}{x+2}$ on the interval $[1,2]$ is the slope of the line that passes through points $P(1, f(1))$ and $Q(2, f(2))$. What occurs with the instantaneous rate of change of $f(x)$ at the point $(1, f(1))$?

(b)

Some values of the continuous function f can be found in the following table:

x	0.9	0.99	0.999	0.9999	0.99999	1	1.00001	1.0001	1.001	1.01	1.1
$f(x)$	-2.1	-2.01	-2.001	-2.0001	-2.00001	-2	-2.00002	-2.0002	-2.002	-2.02	-2.2

x	1.9	1.99	1.999	1.9999	1.99999	2	2.00001	2.0001	2.001	2.01	2.1
$f(x)$	3.61	9.9601	3.996001	3.99960001	3.99996	4	4.00004	4.00040001	4.004001	4.0401	4.41

a) Use the table to approximate the value of the derivative of f at $x = 2$.

b) From this information, do you think that $f(x)$ is differentiable at $x = 1$?

(c)

1. Which of the three options corresponds to the learning objective that Simón intends his students to meet? Formulate a response that justifies your selection, considering the different meanings of the concept of the derivative, summarized by Sánchez-Matamoros.
2. Explain why the other two options do not correspond to the learning objective that Simón intends his students to meet.

Figure 1. Task 1, problem 1 (Source: Authors' own elaboration)

(ii) *search for responses,* where the teachers, organized in groups, immerse themselves in the activity find solutions to the problem questions while the TE accompanies the groups in their search, and

(iii) *general discussion,* where a general discussion is carried out based on the responses previously obtained (raw material) with the goal of producing common work in which all "unify their efforts and skills and capitalize on their multiple differences and perspectives" (Radford, 2021, p. 135).

THE TASKS OF THE ACTIVITY

As mentioned, our design considers three tasks whose resolution implies the use of theoretical information regarding different meanings and levels of understanding of the derivative at a point (Sánchez-Matamoros, 2014). In the following sections, we describe in detail these tasks with their respective problems and questions.

Task 1

To get the teachers to establish relations between different problems concerning the derivative at a point and the analytical, analytical-numerical, and geometric meanings of this concept (goal 1 of the design), we produced the following problem.

Problem 1.1

Problem 1.1 (**Figure 1**) revolves around Simón's need to verify that his students have understood the analytical meaning of the derivative at a point after a series of classes oriented towards achieving learning objective Nº 3 (LO3) of the LDI course syllabus for the third and fourth years of secondary education (MINEDUC, 2021). The problem statement begins with Simón's selection of three problems concerning the derivative in which different meanings of this concept underlie. On the one hand, part a in **Figure 1** shows a problem from the *geometric perspective,* based on the idea of the slope of line tangent to a function f at a given point.

On the other hand, part b in **Figure 1** illustrates a problem from the *analytical perspective,* that emphasizes the idea of an incremental ratio (or rate of change) between the increment Δy of the function f on a certain interval Δx. Finally, part c in **Figure 1** shows a problem from the *analytical-numerical perspective* under the fundamental idea of approximating a rate of change that requires estimating the value of the derivative of f at a point in its domain.

Starting from the problems chosen by Simón, we pose two questions that invite teachers to work collectively on identifying the problem that best fits LO3, justifying their choice regarding the characteristics of the three meanings of the derivative provided by Sánchez-Matamoros (2014). With these questions, we seek to create conditions for teachers to critically position themselves regarding the possible relations between these problems and the meanings they establish.

As an example of an anticipated response to the problem, we pose the following hypothetical explanation to the first problem by a group of teachers:

> As we know that each meaning is associated with an understanding of the concept [of the derivative] in and of itself, and since the objective [LO3] seeks the modeling of situations that involve instantaneous rate of change, the only one of Simón's three options that is oriented towards the rate of change is letter (b) because it is precisely there that he asks himself what happens with the instantaneous rate of change of the function at point $(1, f(1))$.

This hypothetical response highlights the relation that teachers establish between the analytical meaning of the derivative that underlies LO3 of the course syllabus and certain mathematical elements required by these problems, and that are specific to each corresponding meaning of the concept. In the response, attention is focused on the phrase "instantaneous rate of *change*" (a characteristic aspect of LO3), implying that there is an incremental ratio in the function that is brought to the activity through the question "What occurs with the instantaneous rate of change of $f(x)$ at the point $(1, f(1))$?" In this way, teachers have the conditions to conclude that, given Simón's purpose, problem (b) is the one that corresponds to LO3.

Task 2

To promote the establishment of relations between the students' answers to problems concerning the derivative at a point and the intra, inter, and trans levels of understanding of this concept (goal 2 of the design), we produced three problems that we detail below.

Problem 2.1

Problem 2.1 (**Figure 2**) is situated at a moment after an assessment that Simón applied to his students, to ensure the progress of LO3 in the subject he teaches. The problem statement mentions that Simón used the problem in part b in **Figure 1** (part a in **Figure 2**) for this evaluation. Problem 2.1 also includes two student responses[3] (part b and part c in **Figure 2**) that reveal different levels of understanding of the concept of the derivative associated with the problem used by Simón. Based on these responses, the teachers are invited to work collectively to:

(i) score the students' responses according to a given numerical scale, in which the correspondence of these scores with the intra, inter, and trans levels of understanding of the derivative is left to the discretion of each group of teachers and

(ii) propose pedagogical actions that favor the advancement of students in their level of understanding of the concept of the derivative and, consequently, the improvement of their scores.

Through these questions, we seek to create conditions for teachers to position themselves critically regarding their points of view on the two questions contained in problem 2.1.

The theoretical information provided to the teachers to answer the previous questions includes a characterization of the intra, inter, and trans levels of understanding, considering the mathematical elements manifested in the students' responses and the relations between these elements. An example of an expected response to the problem is the following hypothetical explanation from some teachers to the first question:

> One can see that both students solve the problem in a very similar manner and that neither responded to the problem question in a direct form, concerning the instantaneous rate of change. So, if we assign 2 points for each level of understanding, we think that Héctor has 1 point because, being on the Intra level, he justifies his response by specifying the steps taken to determine the average rate of change without including a comparison with the slope of the line that passes through points P and Q. For her part, Javiera would have 3 points because, being on the Inter level, she begins to establish a relation between the average rate of change and the slope of the line that passes through points P and Q, but without including its relation to the instantaneous rate of change.

One of the important aspects in this response is the correspondence that is established between the points scale and the intra, inter, and trans levels of understanding of the concept of the derivative, with the aim being to assign a score to each student. In this response, the teachers pay attention to the mathematics elements necessary to solve the problem (average rate of change, slope of the line between two points and instantaneous rate of change) and the relations between these elements, established (or not) in the students' responses. Note that image *a*, *b*, and *c* in **Figure 2** have been taken and adapted from Sánchez-Matamoros (2004).

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[3] These responses were extracted from Sánchez-Matamoros (2004).

Simón implemented an assessment to monitor the progress of his students regarding the achievement of objective LO3*, using the problem shown in Image *a*.

Verify that the average rate of change of $f(x) = \frac{x+3}{x+2}$ on the interval $[1,2]$ is the slope of the line that passes through points $P(1, f(1))$ and $Q(2, f(2))$. What occurs with the instantaneous rate of change of $f(x)$ at the point $(1, f(1))$?

Image a: Problem selected by Simón for the assessment.

In addition to solving this problem, the students had to provide a justification for what they had done. After reviewing the responses, Javiera (Image *b*) and Héctor (Image *c*)[4] provided responses that called Simón's attention.

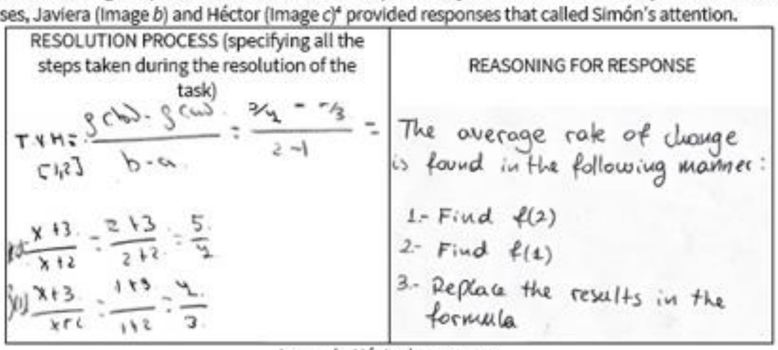


Image b: Héctor's response

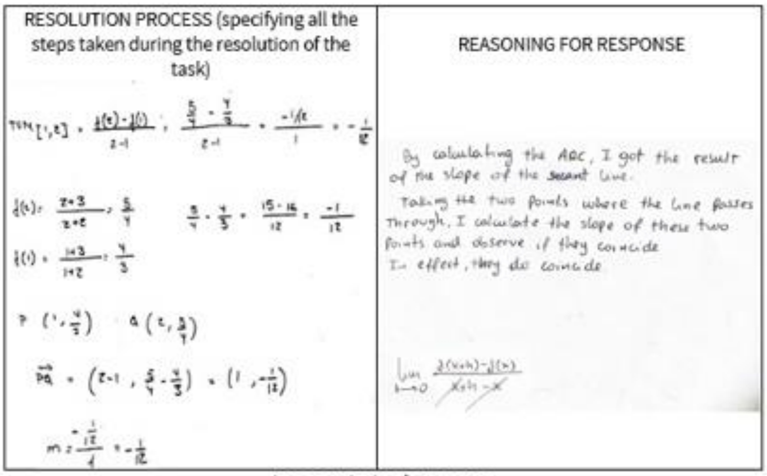


Image c: Javiera's response

1. Assign a score (between 1 and 6 points) to each of Javiera and Héctor's responses, considering the levels of understanding of the concept of the derivative proposed by Sánchez-Matamoros. Justify the score you assigned to each student.
2. What would you do to help Javiera and Héctor to reach high levels of understanding concerning the concept of the derivative and, in this way, achieve a better score?

(*): "Model situations or phenomena that involve instantaneous rate of change and evaluate the eventual necessity of adjusting the obtained model" (MINEDUC, 2021, p. 29).

Figure 2. Task 2, problem 1 (Source: Authors' own elaboration)

Problem 2.2

Problem 2.2 (**Figure 3**) poses a new situation in which Simón intends to achieve a learning objective different from LO3. Specifically, this teacher seeks to create conditions such that his students understand the geometric meaning of the derivative of a function f at a point. The problem 2.2 statement indicates that Simón carried out a new evaluation of his students, consisting of solving two problems. Part a in **Figure 3** shows one of these problems in which the graph of a function f and the equation of line L tangent to f at a specific point are given. In this context, students must evoke the geometric interpretation of the derivative at a point (mathematical element) to recognize that the value of the derivative of f at $x = 2$ coincides with the value of the slope of line L. For its part, part b in **Figure 3** illustrates one of the responses[4] to this problem in which a student does not explain the expected mathematical element and only limits himself to using an analytical procedure to determine $f(2)$ and $f'(2)$, without necessarily considering the graphical aspects of the problem.

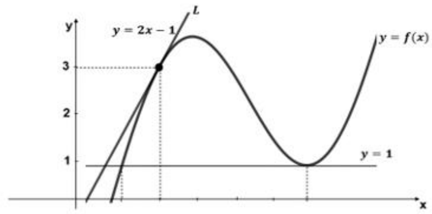
Based on the student's response, the problem invites the teachers to encounter each other to assign a score to his work and explain the reasons for the assigned score. In this way, we seek to make the teachers feel the need to support their decisions at the intra, inter, and trans levels of understanding of the concept of the derivative and position themselves critically with respect to the score assigned to the student.

Below, we present the following hypothetical response to problem 2.2:

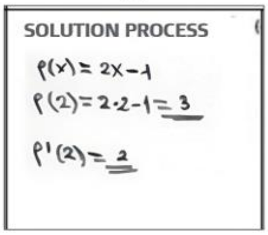
[4] This response was extracted from Sánchez-Matamoros (2004).

Suppose that Simón's objectives change. Now, he considers it appropriate that his students come closer to the *geometric meaning of the derivative of a function f at a point.* To monitor the progress of this learning objective, Simón implements an assessment with two problems. Image *a* shows the first of these problems and Image *b* shows one student's (Fernando) response to this problem.

Suppose that line L is tangent to the graph of the function at point (2,3) as shown in the figure. Find $f(2)$ and $f'(2)$.



(a)



(b)

Assign a score to Fernando's response to the previous problem and explain why you assigned this score.

Figure 3. Task 2, problem 2 (Source: Authors' own elaboration)

> It was difficult to assign a score to the student because he [Fernando] arrived at correct answers, but through incorrect or improper methods. In the case of $f(2)$, it is likely that Fernando substituted $x = 2$ in the equation of the line tangent to function f because this is the only [equation] present in the problem. In the case of $f'(2)$, the student limits himself to writing the result, without explaining the resolution procedure. We believe that Fernando has 2 points because, being at the Intra level, he does not make explicit that he used the provided graph to arrive at the answer. In other words, there is nothing that tells us that Fernando understands that $f'(2)$ is the slope of the line tangent to the given point.

In this hypothetical response, it is striking how difficult it seemed to be to assign a score to what Fernando did, due to the discrepancy between the correct answers provided by this student and the incorrect and non-explicit methods he applied. It is in this instance that the levels of understanding of the concept of the derivative play a fundamental role, in the sense of serving as a reference to the teachers when assigning a score more in accordance with the type of response provided by the student.

Problem 2.3

Problem 2.3 (**Figure 4**) revolves around Simón's need to know what Fernando (his student) thought about the meaning of the derivative of f when formulating his answer in part b in **Figure 3** (see problem 2.2). The problem statement explains that this need led Simón to start a conversation[5] with his student (part b in **Figure 4**) to clarify the procedures he used in his response. This conversation reveals that Fernando does not understand the geometric meaning of the derivative at a point, understood as the slope of the line tangent to function f at the value $x = 2$ (mathematical element). Based on this information, problem 2.3 invites teachers to work collectively to:

(i) recognize the type of reasoning that Simón expects from the student after talking with him about his response and

(ii) identify the level of understanding of the concept of derivative of the student given the conversation.

An example of an expected response for the second question of problem 2.3 is given in the following hypothetical explanation:

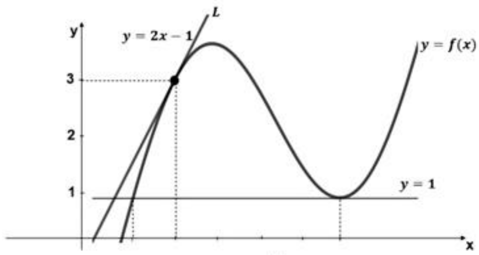
> Unlike the previous problem [problem 2.2], we now understand better how Fernando solved the problem. The conversation with the teacher indicates that the student is at the Intra level [of understanding the concept of the derivative], since he makes it explicit that, to calculate $f'(2)$, he used the expression $2x - 1$ and has derived it. However, when Simón asks him how else he could justify his procedure, Fernando does not use the graph to answer. Therefore, we believe that it is not clear to the student that the derivative of f at $x = 2$ is the slope of the line L. Now, if we had to score his answer again, we would give him 1 point.

In this response we highlight a positioning of the teachers regarding Fernando's explanations that reveals both a better understanding of these subjects of the procedure used by the student to calculate $f'(2)$, as well as the establishment of a relation between the Intra level of understanding of the concept of the derivative and the fact that Fernando did not resort, in his explanations, to the desired mathematical element to solve the problem in part a in **Figure 4**.

[5] This conversation was extracted from Sánchez-Matamoros (2004).

Upon reviewing Fernando's response to the first problem in the assessment (Image *a*), Simón talks with the student to better understand the way in which he solved the problem. Image *b* shows the corresponding conversation between Simón and his student.

Suppose that line L is tangent to the graph of the function at point (2,3) as shown in the figure. Find $f(2)$ and $f'(2)$.



(a)

Simón: To find the value of $f(2)$, you substitute $2x-1$ for the line. Could you find this value in another way?

Fernando: Yes, based on the graph, for $x = 2$, the "y" value is 3.

Simón: In the case of $f'(2) = 2$, what did you do?

Fernando: I derived "$2x - 1$" and it gave me 2.

Simón: In the task, you are asked for $f'(2)$, considering $f(x)$ to be this curve [pointing to the graph of $f(x)$ in Image *a*], from which you don't know the analytical expression. Do you know how to apply another type of reasoning?

Fernando: No, I don't know.

(b)

1. In a moment of the conversation, Simón performs the following intervention:

   Simón: *In the task, you are asked for* $f'(2)$, *considering* $f(x)$ *to be this curve* [pointing to the graph of $f(x)$ in Image a], *from which you don't know the analytical expression. Do you know how to apply another type of reasoning?*

   What reasoning does Simón expect from Fernando and why?
2. When asked this question by Simón, Fernando answers "No, I don't know" (see Image b). In light of this, identify what level of understanding the student is at.

Figure 4. Task 2, problem 3 (Source: Authors' own elaboration)

Task 3

For teachers to position themselves critically regarding the usefulness of the intra, inter, and trans levels of understanding in the analysis of interactions of students who work together in solving derivative problems (goal 3 of the design), we produced problem 3.1, described below.

Problem 3.1

Problem 3.1 (**Figure 5**) takes place at the moment when Simón asks his students to collectively solve the second problem[6] of the evaluation outlined in problem 2.2. Problem 3.1 is accompanied by the following materials:

(i) a conversation held by three students (Sofía, Daniela, and Nicole) from the same group, when solving the problem proposed by Simón (see **Appendix A**) and

(ii) the response that this group gave to the teacher (see **Appendix B**).

With this information, we invite teachers to assign one of the levels of understanding of the concept of derivative to what the students have done, considering the information contained in the problem materials. It is worth noting that the difference in the assignment of the level of understanding, with respect to what was done in the previous problems, lies in the fact that the answer provided in **Appendix B** was produced collectively, which would lead teachers to question the assignment of the same level of understanding to the three students in the group separately.

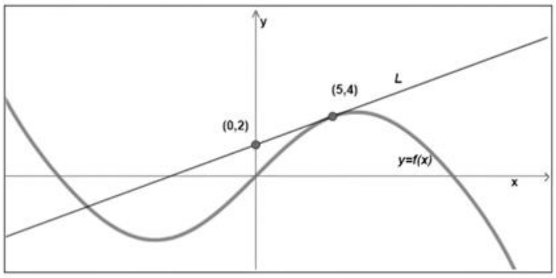
As an example of an expected response by the teachers given this problem, we pose the following hypothetical response by a group of teachers to the first question:

> When reading the materials, it is clear to us that the students had difficulties solving the problem. It was difficult for us to decide at what level [of understanding of the concept of the derivative] the group's response is. However, we agreed that this answer is at the Intra level, because the students seem not to recognize that the slope of the line L given the points [mathematical element: average rate of change] coincides with $f'(5)$, as Nicole shows in the conversation when she asks: "And then what do we do with that slope?" [see **Appendix A**]. The students also do not seem to understand that the derivative of f is the slope of the tangent line to the function at $x = 5$ [mathematical element]. We see this when they derive $f(x) = \frac{2}{5}x + 2$ and are surprised when they cannot replace $x = 5$ in the [obtained] result.

[6] Originally, this problem was employed by Sánchez-Matamoros (2004) with 16-17 year old students who responded to the problem individually.

Image *a* shows the second problem in the assessment implemented by Simón to monitor the learning of the concept of the derivative of a function f at a point from the geometric perspective. Conscious of the difficulty of this problem with respect to the previous problem, the teacher invites the students to solve it in teams.

Given the following figure:



Suppose that the line L is tangent to the graph of the function at point (5,4) as is shown in the figure. Find the values of $f(5)$ and $f'(5)$. Write your response in the box below and explain how you got each answer.

(a)

After a few minutes, Simón visits a team to check on their progress in solving the problem. At first, he listens to the conversation between Sofía, Daniela and Nicole (see Appendix 1).

1. Assign a level of understanding to the response given by the team (see Appendix 2).
2. To what extent do Sofía, Daniela, and Nicole find themselves on the level assigned to their response? Explain.

Figure 5. Task 3, problem 1 (Source: Authors' own elaboration)

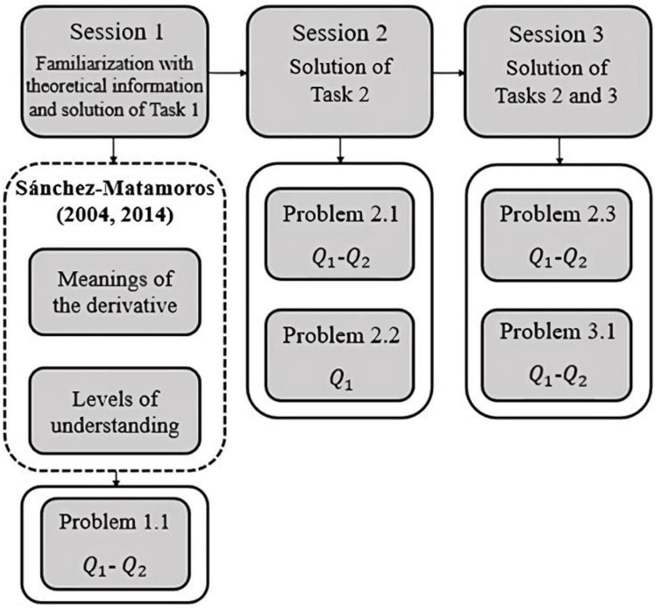


Figure 6. Implementation structure (Source: Authors' own elaboration)

An important feature of this response is the progress that is evident in assigning a certain level of understanding of the concept of the derivative at a point to a given response. However, we developed problem 3.1 with the purpose of generating conflict in teachers regarding assigning a level of understanding to a response that has been produced collectively, when such levels respond to individual responses.

IMPLEMENTATION OF THE TASKS

The previously described tasks are expected to be implemented with mathematics teachers that work in educational establishments in the northern region of Chile and that find themselves teaching (or have taught) the LDI course of the differentiated curricular plan in these establishments. Implementation is intended to be carried out in the form of a certified in-person *workshop*, organized in three sessions lasting 90 minutes each (**Figure 6**).

Sessions 1 will be divided into two parts. In the first part, teachers will become familiarized with the theoretical information related to the different meanings and levels of understanding of the derivative at a point that Sánchez-Matamoros (2004, 2014) proposes. To this end, the sessions will begin with a discussion on the material conditions of teaching the concept of the derivative that teachers experience in the establishments where they work in order for these subjects to position themselves critically regarding the possibilities and limitations of their work with this content in the classroom. This positioning will create conditions that give rise to the need to conscientiously use the theoretical information discussed in the session to judiciously select derivative problems and meaningfully evaluate their resolution by the students. In the second part, teachers will be invited to solve task 1, corresponding to the description of the Θ component of our design. In the same way, sessions 2 and 3 will be dedicated to the resolution and discussion of task 2 and task 3.

In each session, it is anticipated that the teachers and the TE commit to making the workshop a space of encounter with other voices in which they "can encounter cultural knowledges and voices in deep conceptual ways while at the same time making the experience of collective life, solidarity, plurality, and inclusivity" (Radford, 2021, p. XII). With regard to the TE, we understand that this subject plays a key role in the implementation of the tasks, in the sense that he becomes part of a collective consciousness in the search for shared answers to the proposed problems, creating conditions for genuinely collective work with the teachers. In this way, the TE will intervene to offer the teachers alternative ways of addressing the answers in the tasks, such that these subjects consider them from their emerging perspective (Radford, 2021).

FINAL REFLECTIONS

In this article, we describe the design of a training activity oriented towards the encounter of mathematics teachers with historical-cultural forms of thinking regarding the analytical, analytical-numerical, and geometric meanings of the derivative at a point, as well as the intra, inter, and trans levels of understanding proposed by Sánchez-Matamoros (2004, 2014).

We put together this training activity with two purposes in mind. First, with our design we seek to contribute to overcoming the partial vision of the concept of the derivative (more analytical) that underlies LO3 in the LDI course syllabus, offered during the final years of secondary education in Chile. Next, the design of the activity responds to the need to rely not only on orientations regarding how to present this content in the classroom, but also on theoretical information that strengthens the views of mathematics teachers regarding the way in which students in their final two years of secondary education reflect, act, and think about the derivative when they solve problems related to this concept.

To carry out the design, we rely on the concepts of learning and activity from the TO perspective. With respect to learning, we adopt a point of view of this phenomenon as the progressive encounter with historical-cultural forms of thinking that allows teachers to *become aware* of the meanings and levels of understanding of the derivative at a point and the co-production of these subjects as ethical and responsible mathematics teachers regarding this knowledge. In this vision of learning, "the teacher understands that it is necessary to take a critical look at the proposals coming from government spheres and present opinions on the recommended curricula" (Vargas-Plaça & Radford, 2023, p. 7). Methodologically, our design was inspired by the Φ and Θ components of any teaching-learning activity. From the Φ component, we designed the activity based on the object-goals-tasks structure, formulating three goals to which we assigned one task each. Through the Θ component, we propose the moments of the activity through which we project that the teachers and the TE pass while thinking about learning the derivative.

Finally, we consider our design to be a contribution to the field of mathematics education and, particularly, to those researchers on the mathematics teacher education who are beginning to base their studies on theoretical/methodological assumptions of the TO. We believe that working with in-service teachers provides us with a favorable context for the constitution of a collective consciousness about teacher education processes in mathematics, enriched by the experience of teachers in the classroom and strengthened by the theoretical information that is made available in design tasks. Therefore, our commitment to this work lies in the implementation of the design and the subsequent analysis of the data produced from this implementation, thus ensuring the continuity of research in the future.

Author contributions: ABP-A, JLP-G, & REG-A: conception and design of the manuscript & **ABP-A:** manuscript idea, literature review, & the first draft. All authors agreed with the results and conclusions.

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Ethical statement: The authors stated that the study does not require ethics committee approval. It is the design of a training activity for mathematics teachers that has not been implemented.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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APPENDIX A: CONVERSATION BETWEEN DANIELA, SOFÍA AND NICOLE

Daniela: *I believe that … since they gave the two points* [indicating points $(0, 2)$ and $(5, 4)$ on the graph], *I believe that we have to find the function of the line* [its analytical expression].

Nicole: *It's just that what we are interested in is this* [pointing to the graph of $f(x)$]. *Not the line.*

Sofía: *If we have the equation* [of the line], *we can replace* $x = 5$ *in the function* [referring to the analytical expression of the line] *and get* [a value of] y *in the same way.*

Nicole: *What we need to know is the equation* [of $f(x)$] *… I don't know.*

Sofía: *Let's calculate the slope* [of line L] *then.*

Nicole: *And then what do we do with that slope?* [she asks between laughs].

Sofía: *Find the function of this* [indicating line L] *… and that would be* 0.4 [referring to the value of the slope of line L] … [the students dedicated a few minutes to determining analytically the equation of line L]. *So,* [the equation] *would be* $\frac{2}{5}$ *of* x *plus* 2.

Daniela: *Now, we express* [the equation] *as a function* [writing "$f(x) = \frac{2}{5}x + 2$"].

Sofía: *Now, we have to derive this* [function].

Nicole: *So that would leave us …* [they derive $f(x)$ obtained from the equation of the line].

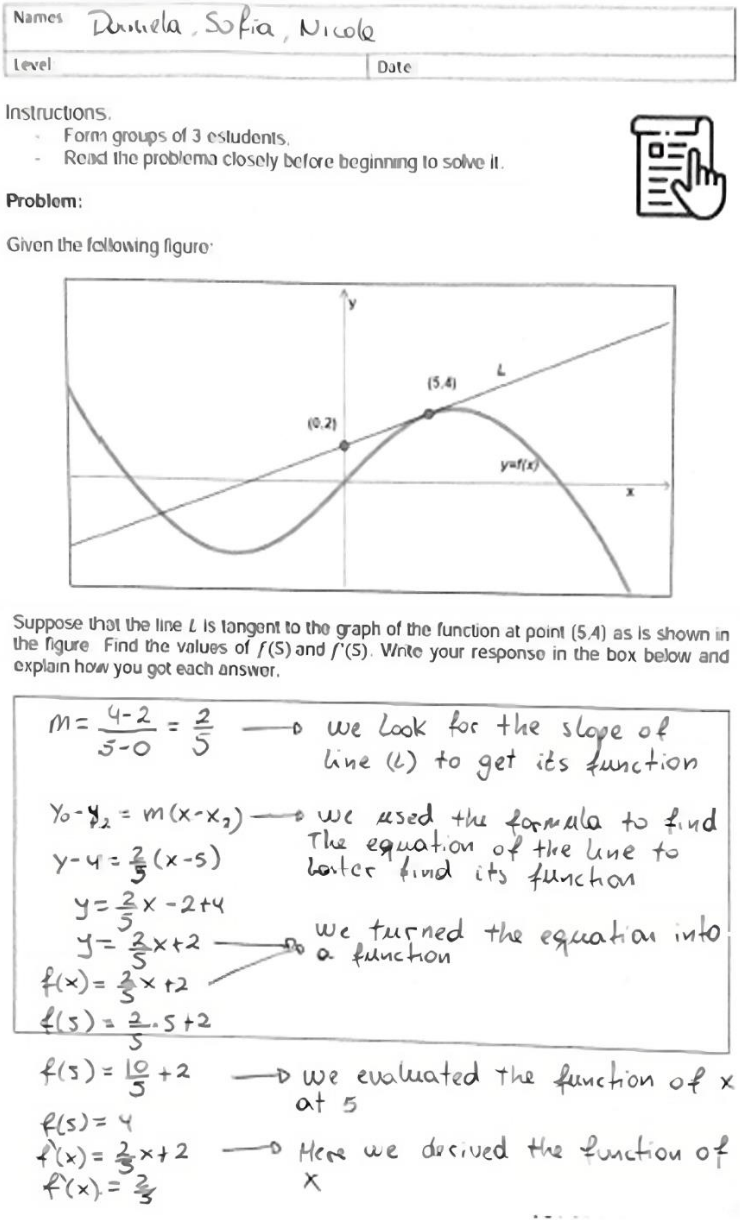
Sofía: $\frac{2}{5}$. *And where do we replace the* 5*?*

Daniela: *Yes, there is no* x.

Nicole: *Hey, yes.*

Sofía: *I don't know. I don't know.*

APPENDIX B



Names: Daniela, Sofia, Nicole

Level | Date

Instructions.

- Form groups of 3 estudents.
- Read the problema closely before beginning to solve it.

Problem:

Given the following figure:

Suppose that the line L is tangent to the graph of the function at point (5,4) as is shown in the figure. Find the values of $f(5)$ and $f'(5)$. Write your response in the box below and explain how you got each answer.

$m = \frac{4-2}{5-0} = \frac{2}{5}$ → we look for the slope of line (L) to get its function

$y_0 - y_2 = m(x - x_2)$ → we used the formula to find The equation of the line to later find its function

$y - 4 = \frac{2}{5}(x-5)$

$y = \frac{2}{5}x - 2 + 4$

$y = \frac{2}{5}x + 2$ → we turned the equation into a function

$f(x) = \frac{2}{5}x + 2$

$f(5) = \frac{2}{5} \cdot 5 + 2$

$f(5) = \frac{10}{5} + 2$ → we evaluated the function of x at 5

$f(5) = 4$

$f'(x) = \frac{2}{5}x + 2$ → Here we derived the function of x

$f'(x) = \frac{2}{5}$

(Source: Authors' own elaboration)