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Different Approaches to Mathematical Modelling: Deduction of Models and Student's Actions

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The idea about a general mathematical modelling process is one of the main components of the studies for teaching and learning mathematical modelling. In this article we are interested in showing that the approaches given to mathematical modelling activities, even if they are in perspective related to learning and instruction, can be distinguished, especially as regards the construction and use of mathematical models. To this end we present two activities that differ primarily in relation to data and methods used to obtain the model: in a situation models are obtained an analyzed from qualitative information while in another situation that construction starts from quantitative data about the phenomenon under study. What the students have to do – the students' actions- differ in these different approaches.

Keywords: Mathematical Modelling, Student's Actions, Phases of Modelling

INTRODUCTION

The idea about a general mathematical modelling process is one of the main components of the studies for teaching and learning mathematical modelling. Analytically, the modelling process is described as a cyclic process involving phases and the models are constructed for one or more curricular reasons or to enhance the ability of students to solve authentic real world or life problems.

In this article we are interested in showing that the approaches given to mathematical modelling activities, even if they are in perspective related to learning and instruction, may be distinguished, especially as regards the construction and use of mathematical models. If, on the one hand, quantitative information about a phenomenon underlies the construction of mathematical models to study the problem, on the other hand, a qualitative analysis of a problem situation may also be supported by mathematical models.

Although we have phases for the development of a modelling activity, the actions of students, during these phases, may be essentially different in terms of these different approaches. In order to illustrate these two possibilities for the construction and use of mathematical models in real-world phenomena, and the student's actions in these cases, we have discussed two activities involving environmental issues.

About Mathematical Modelling

Into the Mathematics Education, according to Galbraith (2012), we can identify different approaches to the mathematical modelling. The understanding we have in mind in this text,

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based on Almeida (2010), is that, in general, a mathematical modelling activity may be described in terms of an unfamiliar and problematic situation (initial situation), an expected final situation (which represents a solution to the initial situation) and a set of procedures and actions needed to pass from the baseline to the final situation. The mathematical model, which is the mathematical representation associated with this situation is, according to Lesh et al (2006), a conceptual system, descriptive or explanatory, expressed through a language or a mathematical structure, in order to describe the behavior of another system and allow predictions on this other system. Also according to the authors, it is possible that the model built to represent a situation at any given time serves also to represent another system at a later time.

The construction of models has been considered as one of the strategies with potential to lead to learning, shared vision by Howland, Marra and Jonassen (2011), for which,

> *Humans are natural model builders. From a very early age, we construct mental models of everything that encounter in the world. [...] These models comprise their personal theories about the world that enable them to reason about the things that we encounter. Modelling helps learners express and externalize their thinking, visualize and test components of their theories and make materials more interesting (pp 192).*

Therefore, by introducing modeling activities in mathematics learning, we may develop in the students skills both for building mathematical models and for the interpretation of results obtained with these models.

Abilities and skills to mathematizing real world situations are essential to human development today. Thus, it is important to consider issues of reality as a starting point for modelling activities by setting the activity as something wherein

> *The starting point is normally a certain situation in the real world. Simplifying it, structuring it and making it more precise – according to the problem solver's knowledge and interests – leads to the formulation of a problem and to a real model of the situation. [...]. If appropriate, real data are collected in order to provide more information about the situation at one's disposal. If possible and adequate, this real model – still a part of the real world in our sense – is mathematized, that is the objects, data, relations and conditions involved in it are translated into mathematics, resulting in a mathematical model of the original situation. Now mathematical methods come into play, and are used to derive mathematical results. These have to be re-translated into the real world, which is interpreted in relation to the original situation. At the same time the problem solver validates the model by checking whether the problem solution obtained by interpreting the mathematical results is appropriate and reasonable for his or her purposes. If need be (and more often than not this is the case in 'really real' problem solving processes), the whole process has to be repeated with a modified or a totally different model. At the end, the obtained solution of the original real world problem is stated and communicated (BLUM, 2002, p. 152-153).*

Considering this configuration, we agree that the current view of mathematical modelling stemming from a models-and-modelling perspective (Lesh & Zawojewski, 2007) sees students' modelling process as going through multiple cycles in developing a mathematical model for a given problematic situation.

According to Ferri (2006), these cycles are different, because they are dependent on various directions and approaches of how modelling is understood and in some cases, if complex or non-complex tasks are used. Kaiser and Sriraman (2006) classify approaches and backgrounds of the modelling discussion, and from this discussion it became much more evident, that different standpoints and views on modelling exist.

In this context, our paper will refer to a set of phases characterized in Almeida, Silva and Vertuan (2012) and the actions of students during these phases. According to these authors, the phases are related to the set of procedures required for configuration, structuring and solving of a problem situation and are characterized as: Becoming aware, Mathematisation Resolution, Interpretation and Validation.

Becoming aware: becoming aware of something refers to "being familiarized with", "collecting information about" and to "the act of being aware". In terms of mathematical modelling activity this phase represents a first contact with a problem situation which is intended to be studied in order to know the characteristics and specificities of the situation.

Mathematisation: this phase is characterized by "identifying the relevant mathematics with respect to a problem situated in reality, representing the problem in a different way, including organizing it according to mathematical concepts and making appropriate assumptions, understanding the relationships between the language of the problem and the symbolic and formal language needed to understand it mathematically, finding regularities, relations and patterns, translating the problem into mathematics i.e. to a mathematical model" (PISA 2006, p.96),

Resolution: working accurately within the mathematic world, which includes "using and switching between different representations, using symbolic, formal and technical language and operations, refining and adjusting mathematical models, combining and integrating models, argumentation, generalization"(PISA 2006, p.96).

Interpretation and Validation: interpreting the result and validating a model, includes interpretation of mathematical results in a real solution in the real world, "understanding the extent and limits of mathematical concepts, reflecting on mathematical arguments and explaining and justifying results […], critiquing the model and its limits" (PISA 2006, p.96),

During these phases of modelling the students engage in a set of actions that are characterized here as: the search of information; the identification and selection of variables; the construction of hypotheses; the simplification; the transition of mathematical language; the activation of prior knowledge; the use of techniques and/or mathematical procedures; the comparison and distinction of ideas; the generalization of facts; the articulation of knowledge from different fields; the argumentation to expose to others the judgment of the value of theories and methods used in the development of the activity.

These students' actions, however, may be more intense or less intense according to the type of information, mathematical procedures and interpretations are required to search the solution.

So, in this article we are interested in showing that the approaches given to mathematical modelling activities may be distinguished, especially as regards the construction and use of mathematical models. If, on the one hand, quantitative information about a phenomenon underlies the construction of mathematical models to study the problem, on the other hand, a qualitative analysis of a problem situation may also be mediated by mathematical models. The students' actions may vary as a result of these possibilities for the modelling activity. For the purpose of illustrating how can set these different possibilities, we describe briefly two activities.

THE ACTIVITIES

We present two modelling activities. The first, "A qualitative study for the biological control of an infestation", represents a propose to be developed with students of a course of biology, for example, in order to indicate what actions students and / or teachers could be performed during different phases of mathematical modelling. The second activity, "Global warming as an investigation", was developed by students of a Master Degree course in Mathematics Education during the discipline of Mathematical Modelling in the Perspective on Mathematics Education. In the second activity we present the records done by students which are in the report delivered by them in order to identify their actions.

Activity 1: A qualitative study for the biological control of an infestation

This activity regards to the qualitative study of the biological control of an infestation. An infestation occurs when a particular population suddenly increases its population density and then returns to its previous state. In general, an infestation comes from some environmental disturbance such as sudden change in climate or environmental disaster, but it may also occur due to the own natural cycle of the species. In this context, the construction and analysis of mathematical models may offer interesting solutions to these problems and it can be doing by students of courses in the biological area.

The phase becoming aware is the identification of a situation where the control for biological pest can be identified. The most relevant action by the students at this phase is the search for information in order to understand the situation.

We are interested in showing how a mathematical model for the study of biological control of an infestation may be constructed from qualitative information of the problem and, by using the classical models of population dynamics. The mathematical analysis of the model involves the basic theory of Differential Equations and other mathematical tools of Differential and Integral Calculus, seeking qualitative information that assists decision-making for the biological control of this pest. Thus, this is the mathematisation in this activity.The resolution phase requires an analysis of different features of the problem and implies the resolution of different kinds of differential equations. The answer to the problem comes from the qualitative analysis of the solution.

In general, to examine the variation of the population density of a particular invasive species in a given region, we indicate N=N(t) as the number of individuals at time t, where t is a real variable, which means we are able "to estimate" the value of N(t) in any fraction of time. In general, in a course of calculus differential or in course of differential equations this problem can be analyzed by different approaches:

1- We may represent population dynamic through the Malthus model - In this case, the model indicates that the infestation would remain for an indefinite period and with the number of individuals increasing exponentially.

2- However, there are several natural factors that limit this population growth. Thus, it is reasonable to assume that there is a maximum amount of individuals, we might presume it is constant, which is supported by the environment and, besides this capacity, the population naturally declines due to lack of food. This condition is described in the Verhulst model.

3- The logistic model, although it represents a more realistic long-term population dynamics, is not sufficient to study the control of an infestation. One way to control an infestation would be to introduce a factor of predation to the invasive species. Then the obtained model is:

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - p(N)$$
$$N(0) = N_0$$

In this case, we can consider:

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - \frac{\beta N^2}{\alpha^2 + N^2}$$
$$N(0) = N_0$$

The predation function, $p(N) = \frac{\beta N^2}{\alpha^2 + N^2}$, represents the dynamics of this pest elimination by a species of birds, since food will be readily available, which means that there is no predation during periods of low pest density. This phenomenon which may be mathematically interpreted by fact that $\lim_{N\to 0} p(N) = 0$. Another feature of this predation is that its effect has a saturation level for high densities of pests, which is represented by the term β in equation p(N). Thus, we have $\lim_{N\to\infty} p(N) = \beta$.

In Figure 1 we illustrate the p(N) curve where we are able to see that $(\alpha, p(\alpha))$ is an inflection point indicating the moment in which this saturation begins to occur. The concepts of equilibrium and stability are related to the lack of variation in a system, and in problems involving dynamic events such as growth enable us to understand certain behaviors of a system better.

By making $\frac{dN}{dt} = 0$ in (1) we have: $\frac{dN}{dt} = 0 \Leftrightarrow rN\left(1-\frac{N}{K}\right) - \frac{\beta N^2}{\alpha^2 + N^2} = f(N) = 0$

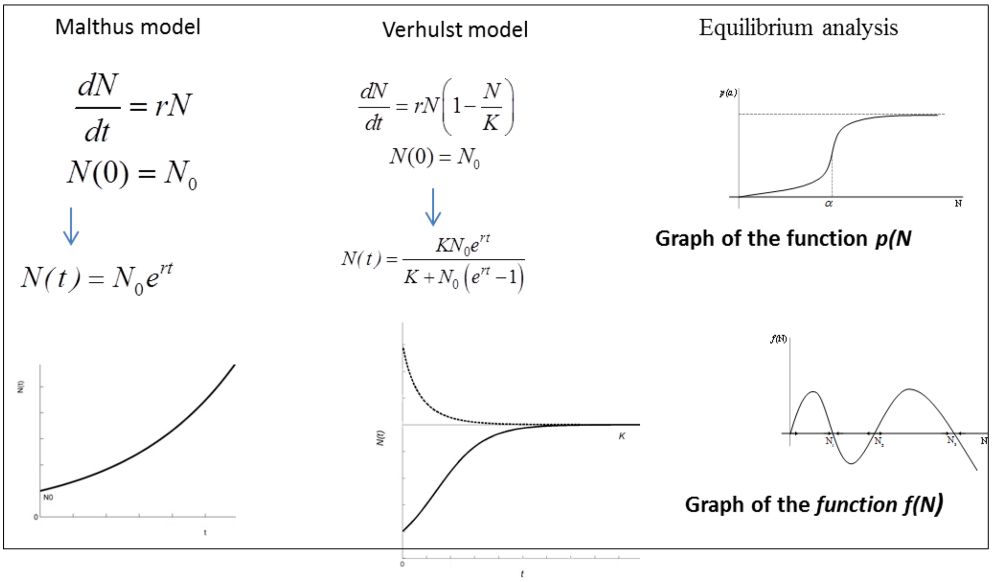

Figure 1. Models for the study of biological control of an infestation

The figure 1 shows one of the possibilities for the graph of the function f in which we have four values for N for which $f(N)=0$. As f is increasing for N equal to 0 and N2 as well as decreasing for N1 and N3 we have, that 0 and N2 represent unstable densities, and N1 and N3 stable densities of the pest population.

Activity 2: Global warming as an investigation

This activity was developed by a group of students in the discipline of Mathematical Modelling in a Masters course in mathematics education. Four students were interested to the theme "Is global warming a problem?" In this case the group proposed to obtain mathematical models which identify the possible options for the confrontation of the problem based on information and data from the IPCC[1] report.

In the essay *Carbon Our challenge - to tackle global warming the first step is to do the math*"[2], published by *National Geographic* magazine in October 2007, Bill McKibben points out that reports have been published in the last years suggesting that 450 ppm (parts per million) as the threshold limit value for the concentration of CO_2 in the atmosphere. Read that reporting was the motivation of students to start modelling activity.

After this report, the students have concentrated their efforts to search for pieces of information that deal with the variation of the global concentration of CO_2 in the atmosphere and with the variation of the global temperature over the years. They have obtained on the Carbon Dioxide Information Analysis Center (CDIAC) website data on the average concentration of CO_2 in the atmosphere (in ppm), and on the Earth Policy Institute website they could find some data on the annual average temperature of the planet (in degrees Celsius). Other websites supply information to that respect.

Thus, the 'Becoming aware' phase in this activity consists in obtaining information in magazines and websites, discussion with classmates and with the teacher in order to finding out important aspects of a mathematical approach to the problem of global warming.

Based on this information the students have invested in developing models to investigate the questions: If the growth trend of the concentration of CO_2 remains in the pace it has

[1] IPCC: Intergovernmental Panel on Climate Change. Report available at http://www.ipcc.ch/pdf/reports-nonUN-translations/portuguese/ar4-wg1-spm.pdf.

[2] Subtitle of the essay published by *National Geographic* magazine in October 2007.

been during the last decades, when will the average concentration of CO_2 in the atmosphere reach 450 ppm? When reached 450 ppm, what would the average global temperature be? And one century from now, what would the average global temperature of our planet be according to the models obtained?

Although the students have found a great deal of information, we limit ourselves here to present the table with quantitative data obtained by the students and articulate how students, with the use of these data, obtained mathematical models to answer research questions.

The mathematisation in this activity, associated with actions such as moving between languages and identify variables, define hypotheses would give mathematical meaning to the organization's reality.

Table 1. Data obtained by the students

Period	i	Concentration of CO_2 $C(i)$(in ppm) (average)	$\left(\frac{C(i+1) - C(i)}{C(i)}\right)$	T_i
1965-1969	0	322.250	0.0175	13.914
1970-1974	1	327.878	0.0184	14.002
1975-1979	2	333.910	0.0224	14.026
1980-1984	3	341.390	0.0230	14.254
1985-1989	4	349.274	0.0204	14.274
1990-1994	5	356.414	0.0225	14.316
1995-1999	6	364.444	0.0244	14.476
2000-2004	7	373.324	0.0175	14.582

Considering the data from the tables, students define the variables: C: average concentration of CO_2 in the atmosphere (in ppm), and i: time intervals (5 years) as suggested in Table 1..

In the resolution phase, the students worked with different hypotheses for the variation in global temperature, thus obtaining different models based on these hypotheses.

Hypothesis 1: The variation of the average concentration of CO_2 in the atmosphere is proportional to the average concentration in each year (table 1).

Resolution: The model: $\frac{dC}{di} = bC$ The solution: $C(i) = 321{,}115.e^{0{,}0211i}$

Analysis in relation to the concentration of 450 ppm they obtained $i = 16{,}056$, which, considering the data in table 1, leads them to infer that this threshold limit value will be reached by the year 2049.

Hypothesis 2: The average temperature of the earth (which from 1965 until 1969 was 13.914 and in the period 2000 to 2004 was 14.582 degrees) will not become infinitely large, that is the earth's temperature is limited, so will stabilize around some finite value, that is, there is a real number T^* such that $\lim_{i\to\infty}(T_i) = T^*$

Resolution: to determine his value of stability the students used Ford-Walford[3] method and obtained the point of stability T*=19,444; analysing the difference between the observed values Ti and the point of stability T* over the years they obtained the model 2: $T_i = 19{,}444 - 5{,}550.e^{-0{,}018i}$

The analysis: to determine the global average temperature when 450 ppm of CO_2 in the atmosphere are reached, which according to the model 1 would occur in *i* = 16 so the students could write $\mathrm{T}_{16} = 19{,}444 - 5{,}550.\mathrm{e}^{-0{,}01816} = 15{,}283$. This way, when 450 ppm of CO_2 in the atmosphere is reached, according to the model, the prediction for the global mean temperature shows that in 2049 the global average temperature would be about 15,283 °C. Ever to consider the issue “And one century from now, what would the average global temperature of our planet be according to the models obtained?”, using the model 2 the students obtained that the global average temperature would be about 16,091 °C in the year 2107.

CONCLUSIONS AND DISCUSSION

Students’ Actions in the Phases of Mathematical Modelling

The analysis we present for the activity 'A qualitative study for the biological control of an infestation' are indications that we do for teachers interested in developing such activities with their students in disciplines of Differential and Integral Calculus or Differential Equations.

In this case, there is no quantitative data from which students can make their formulations models. Considering that, on the theoretical point of view, already are consolidated the indication of models with differential equations suitable for studies of control infestations, the actions of the students focus more on interpretative analysis of the solutions obtained. Thus, in the phase becoming aware the comprehension of how to work a control problem in an infestation is a strong point, even if students are interested in the search of a real situation. This may be mediated by the teacher's instructions and by information searches in literature specialized.

The action, in the phase of mathematisation, in the first activity associated with the identifying the relevant mathematic with respect to a problem situated in reality, likely focus on analyzing about how the hypotheses of models of Malthus and Verhulst are adaptable to the study for the control over a infestation. So, activation of prior knowledge and the transition of mathematical languages are actions that can guide students in this phase.

The resolution of differential equations, relatively simple, may not be the most significant action of the students in the resolution phase. The relevance is in the analysis of parameters of the solutions as well as in their graphic representation by using a software.

It is in the interpretation phase of the solution to the problem, that the actions of the students are important to the construction of the answers. At this phase the comparison and distinction of ideas, the generalization of facts, the articulation of knowledge in different areas, the argument to expose to others the judgment of the value of theories and methods used in the development of the activity confirm the resolutions made by the students.

In the second activity, ‘Is global warming a problem?’, the configuration of the actions of students is different. The students themselves chose a problem to investigate and, at the phase of becoming aware, the action of search of information was a key to success of the activity. The great deal of information regarding the subject, however, would have to be refined, and a specific problem whose resolution they could obtain, needed to be defined. Thus, during the phase of mathematisation the identification and selection of variables,

[3] The method is described in Bassanezi (2002).

simplifying, the transition from representations, the articulation of knowledge from different fields, is who conducted the development of the activity.

The data, in the second activity, were different in nature from the first activity. Many texts, many tables, many indexes had to be analyzed to select information that were relevant and could lead to a study about the problem of global warming. It was necessary to select and, in fact, define hypothesis from the available data. We can infer, therefore, that the efforts of the students in the phase of mathematisation were more intense in this activity. In the resolution phase the student's actions such as activating of prior knowledge, the use of techniques and/or mathematical procedures, the comparison and distinction of ideas and the generalization of facts, led to the development of models.

The interpretation and validation phase, in the second activity was oriented by actions that seek answers to the questions that the students themselves had presented in the previous phase. So, if on the one hand it was necessary to articulate the two models and find the values for average temperature, on the other hand it was necessary to compare the responses obtained with information that they had found in their research, in the phase of becoming aware. That is, the argument to expose to others the judgment of the value to theories and methods used in the development of the activity was an important action in this final phase.

Final Remarks and Implication for Research

The reconstruction of students' modelling processes can be found in many different studies within the literature of mathematical modelling. Many of these studies refer to different cycles or even different modelling perspectives for modelling in mathematics education. However, one must consider that although the cycle and / or the perspective that one has in mind when developing the activity, specificity may result from different kinds of data used and the problem that he/ she wants to develop.

In the first activity the qualitative pieces of information on the problem and the mathematical knowledge of students and/or teachers lead to the use of models already known in literature and the solutions to the problem arise from an analysis of solutions of these models, either in algebraic solutions or in graphical representations of these solutions.

In the second activity the students presented a mathematical treatment of the situation from an analysis of quantitative data about the problem and, from this point on, using some mathematical knowledge, they were able to build mathematical models that supported the analysis of the global warming problem.

Thus, different approaches to mathematical modelling may lead to different actions of the students and it is up to the teacher to define what it expects students learn to organize classes with mathematical modelling.

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