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DIDACTIC EFFECTIVENESS OF MATHEMATICAL DEFINITIONS THE CASE OF THE ABSOLUTE VALUE

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ABSTRACT. Quite often a mathematical object may be introduced by a set of equivalent definitions. A fundamental question is determining the "didactic effectiveness" of the techniques for solving a kind of problem associated with these definitions; this effectiveness is evaluated by taking into account the epistemic, cognitive and instructional dimensions of the study processes. So as to provide an example of this process, in this article we study the didactic effectiveness of techniques associated with different definitions of absolute value notion. The teaching and learning of absolute value are problematic; this is proven by the amount and heterogeneity of research papers that have been published. We propose a "global" study from an ontological and semiotic point of view.

KEYWORDS. Definition, Partial and Holistic Meanings, Cognitive Effectiveness, Absolute Value, Implicative and Hierarchic Statistical Analysis.

MATHEMATICAL EQUIVALENCE VS. DIDACTIC EQUIVALENCE OF DEFINITIONS

One goal for the teaching of mathematics is to channel everyday thinking towards a more technical-scientific thinking at an earlier stage, as a mean for overcoming the conflicts between the mathematics (formal) structure and cognitive progress. Defining mathematical objects involves "more than anything else the conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition" (Vinner, 1991, p.65). This fact justifies the great number of papers in Mathematics Education focussed on mathematical definition (Linchevsky, Vinner & Karsenty, 1992; Mariotti & Fischbein, 1997; De Villiers, 1998; Winicki-Landman & Leikin, 2000; etc.). These researchers refer to specific aspects of the definition: *contexts of use* (geometric, analytical, algebraic, etc.), *mathematical objects* (differentiation, tangent, absolute value, etc.), *properties of the definition*

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(minimalism, elegance, consistency, etc.) *or relations to other mathematical processes* (description, metaphor, model, theorem). We are interested in justifying the large gap between the mathematical equivalence of two definitions of the same object and their epistemic, cognitive or instructional equivalence, that is to say, their didactic equivalence. This theoretical analysis will be carried out for the specific case of the absolute value notion (AVN).

From the viewpoint of Mathematics Education, one fundamental question consists of determining the *didactic effectiveness* of problem-solving techniques associated with a mathematical definition; this effectiveness is assessed by taking into account the *epistemic*[1] (field of applicability of the techniques and mathematical objects involved), cognitive (effectiveness and cost in the use of the techniques by the individuals) and instructional (amount of material resources and time required for its teaching) dimensions. Hence, with the expression *didactic effectiveness* we refer to the articulation of these partial types of effectiveness in an educational project. However, this concept poses us difficult research questions, such as the following:

- Is there a definition that minimises the cognitive and instructional cost of use of resources, maximises the individuals' effectiveness in the specific problem field and facilitates adaptation to new problems?
- Is it possible to classify the definitions according to their scope or generality (field of applicability), their *mutual implication* (one definition may be obtained deductively from another one) or their role *within the institutional practices* (social, cultural, conventional)?

The first step in answering these questions for the case of AVN is determining the nature of this notion and accepting the complexity of objects and meanings that explicitly refer to it. In this article we present a systematic analysis of AVN nature, emphasising the role of its definitions in mathematical work, which is based on the ontosemiotic approach to mathematics cognition (Godino, 2002). Below we present a synthesis of this theoretical framework.

The ontosemiotic approach to mathematics cognition

This is a theoretical framework developed by Godino, Batanero and collaborators for jointly analysing mathematical thinking, the ostensives that support it and the situations and factors conditioning its development. This approach is named ontosemiotic given the essential role it attributes to language and the categorisation of the different types of objects that emerge in mathematical activity. It conceives mathematical language in a wide sense, that includes a variety of expressions and consider as mathematical object any kind of real or imaginary entity to which we refer when performing, communicating or learning mathematics.

This theory starts from the notions about institutional and personal meaning of mathematical objects (Godino and Batanero 1998), where, basing on pragmatic assumptions, they focused on institutional mathematical knowledge, without forgetting the individuals, which

[1] The term 'epistemic' refers to institutional knowledge in this work.

are the focus of educational effort. In that work, they conceive the meaning of a mathematical object (e.g. real number, function, etc.), in terms of the "systems of practices carried out to solve certain types of problems". A practice is "any action or manifestation (linguistic or otherwise) carried out by somebody to solve mathematical problems, to communicate the solution to other people, so as to validate and generalize that solution to other contexts and problems" (Godino and Batanero, 1998, p. 182). The systems of practices are composed by operative (actions, problems) and discursive (notions, propositions, arguments) facets, which are mutually interrelated and complementary. These two facets are articulated by language that allows us to create new mathematical objects and communicate the procedures and meanings involved in mathematical activity.

These operative and discursive practices can either be attributed to individuals – and then Godino and Batanero speak of personal meaning of the object, or be shared in an institution - and then they consider the corresponding institutional meaning. An institutional (personal) object is an emergent of the system of social (personal) practices linked to a problem field.

The theory of institutional and personal meanings is extended in Godino, Batanero and Roa (2005) including the notion of semiotic function and a category of mathematical objects. They propose as primary entities to describe mathematical activity the following:

(1) *Language* (terms, expressions, notations, graphics).

(2) *Situations* (problems, extra or intra-mathematical applications, exercises,...).

(3) *Subjects' actions* when solving mathematical tasks (operations, algorithms, techniques, procedures).

(4) *Concepts*, given by their definitions or descriptions (number, point, straight line, mean, function,...)

(5) Properties or attributes, which usually are given as statements or propositions.

(6) *Arguments* used to validate and explain the propositions (deductive, inductive, etc.).

The main purpose of this article is remarking that the ontosemiotic relativity of mathematical objects and meanings applies both to the institutional and personal frames and to the internal mathematics contexts of use. With this aim we introduce the notion of *partial meaning* for a mathematical object, which is interpreted as the "subsystem of practices" linked to a particular definition thereof. We also introduce the notion of *holistic meaning*, which is understood as the ontosemiotic network made of the different partial meanings. These new theoretical notions will be developed for the particular case of AVN and will allow to progress towards a coherent articulation of the pragmatic and realistic visions of mathematics.

In the following section, a set of research problems are described to show the difficulties of teaching and learning the absolute value notion (AVN). From these investigations we deduce the psychological and pedagogical complexity of the AVN, but none of them tries to integrate the meanings attributed to this notion in the different contexts of use and characterise its ontological and semiotic complexity. Therefore, in relation to the AVN, the first objective of this paper is to show how the different meanings (associated with the contexts of use) are synthesized in different definitions and how these definitions and those meanings can be structured.

Then, we introduce a sample of different definitions of the AVN and, backed by the calculation of the solutions of a linear equation with an absolute value, we indicate how these definitions condition mathematical practices. After, we present a way to structure the objects and meanings associated with the AVN and we describe its holistic meaning. Next, we report a teaching experiment whose purpose is to analyse, from a cognitive viewpoint, the heuristic power of two of the different partial meanings of absolute value identified. Later, we highlight some micro and macrodidactic implications and, finally we provide a brief synthesis and some theoretical implications.

NATURE OF THE NOTION OF ABSOLUTE VALUE

The teaching and learning of the AVN are problematic. This is proved by the amount and heterogeneity of the research papers that have been published. Gagatsis and Thomaidis (1994) describe succinctly the historical evolution of the knowledge about absolute value. They also determine the processes for adapting that knowledge in Greek schools and interpret the students' errors in terms of *epistemological obstacles* (linked to the historical study) and *didactic obstacles* (related with the processes of transposition). More recently, Gagatsis (2003, p.61) argues, on the basis of empirical data that the "obstacles encountered in the historical development of the concept of absolute value are evident in the development of students' conceptions".

From a professional point of view, Arcidiacono (1983) justifies an instruction of the AVN based on the graphic analysis on the Cartesian plane of piece-wise linear functions and Horak (1994) establishes that graphic calculators represent a more effective instrument than pencil and paper for performing this teaching.

On the other hand, Chiarugi, Fracassina & Furinghetti (1990) carried out a study on the cognitive dimension of different groups of students faced with solving problems that involve the AVN. The study determines the need for research that will allow the errors and *misconceptions* to be overcome. On her part, Perrin-Glorian (1995) establishes certain guidelines for the institutionalisation of knowledge about the AVN in arithmetical and algebraic contexts; so she argues the central function of the teacher's didactic decisions in the construction of the AVN, that must take into account the students' cognitive restrictions and must highlight the instrumental role of the AVN.

All these research papers implicitly consider the AVN to be transparent, i.e. they do not see this object as problematic from the epistemological viewpoint. From an ontological and semiotic approach of mathematical cognition and instruction (Godino, 2002; Godino, Batanero and Roa, to appear) it is necessary to theorise the notion of meaning in didactics. These authors start off with the elements of the technological discourse (notions, propositions, etc.) and conclude that the meaning of a mathematical object is inseparable from the pertinent systems of practices and contexts of use.

DEFINITIONS OF ABSOLUTE VALUE

Mathematical definitions and propositions are the most visible part of the “iceberg” that ultimately constitutes the anthropological and cultural reality of mathematics. Mathematical definitions are a discursive component of the systems of mathematical practices, which from an ontogenetic point of view depend on and are “subsequent” to operative practices. But the definitions interact with problems, techniques, and previously established rules in a complex and recursive way, and then pose new questions and new systems of practices.

In this section, we introduce some definitions of the AVN associated with different contexts of use, showing different mathematical texts and describing the relationships that are established between the definitions given. We also briefly indicate how these definitions, as objects emerging from the different subsystems of practices, condition the discursive rules for stating and validating properties and the operative instruments for problem solving, as well as the type of actions and arguments that are acceptable[2]. We exemplified this fact by means of the resolution of a linear equation with absolute value ($|x - 2| = 1$). This equation can be solved of diverse forms. In this section, we shall show how the definitions of absolute value determine specific techniques for resolution. This brief study will allow us to justify the structuring of the objects (definitions, partial meanings, system of practices elements, etc.) associated to the AVN and the relevance of the “holistic meaning” notion.

Arithmetical definition

In the arithmetical context, the AVN represents a rule that *leaves the positive numbers unchanged and changes the negative numbers into positive ones.*

“The absolute value of x, denoted by |x|, is defined as follows:

$|x| = x$ if $x > 0$; $|x| = -x$ if $x < 0$; $|0| = 0$

Thus, the absolute value of a positive number or zero is equal to the number itself. The absolute value of a negative number is the corresponding positive number, since the negative of a negative number is positive.” (Leithold, 1968, p.10).

Then, in order to solve the equation $|\ldots - 2| = 1$, we reason as follows: “the absolute value of a number is 1, then this number is 1 or –1; What number, when subtracting 2 from it, gives 1?, What number, when subtracting 2 from it, gives –1?”. The formalisation of this method may

[2] Another essential question is justifying each definition, that is, finding the systems of operative and discursive practices that have generated the culturally accepted definitions and the language used to represent the networks of objects with which the notion is related. Given the length restriction, we do not enter in this reciprocal question in this article.

be expressed in the following way:

$$|x-2|=1 \Rightarrow \begin{cases} x-2=1 \Rightarrow x=3 \\ x-2=-1 \Rightarrow x=1 \end{cases}$$

Piece-wise function definition

In an analytical context, formally, the AVN is often introduced using a piece-wise function in **Q** and, by extension, in **R**. Then, in order to solve the equation $|x–2|=1$, we reason as follows: In general, the definition of AVN is

$$|(\ )|=\begin{cases} (\ ) & \text{if} \quad (\ )\geq 0 \\ -(\ ) & \text{if} \quad (\ )<0 \end{cases}$$

Then:

$$|(x-2)|=\begin{cases} (x-2) & \text{if} \quad x\geq 2 \\ -(x-2) & \text{if} \quad x<2 \end{cases}$$

Therefore:

$$|x-2|=1 \Rightarrow \begin{cases} x-2=1\,(x\geq 2) \Rightarrow x=3 \\ -(x-2)=1\,(x<2) \Rightarrow x=1 \end{cases}$$

Maximum function definition

The classic definition of absolute value, as a basic notion for the foundations of mathematical analysis, is sometimes reformulated in terms of the maximum function: $|x| = \max\{x; -x\}$. This definition of the absolute value in terms of the maximum function can be extended to complete ordered vectorial spaces in terms of the most general notion of "supreme". In fact, it is possible to define functionally in any ordered field[3] (non empty) the absolute value or, more precisely, the abstract notion of "measurement or norm".

Then, in order to solve the equation $|x – 2| = 1$, since $|x – 2| = \max\{x – 2; –(x – 2)\}$, the proof is reduced to the process performed according to the piece-wise function definition.

Compound function definition

Finally, it is easy to demonstrate that: $|x|=+\sqrt{x^2}$ (Mollin, 1998, p.47). Then, in order to solve the equation $|x–2|=1$, we reason as follows:

$$|x-2|=1 \quad \Rightarrow \sqrt{(x-2)^2}=1 \Rightarrow (x-2)^2=1 \Rightarrow x^2-4x+3=0 \Rightarrow$$
$$\Rightarrow x=\frac{4\pm\sqrt{16-12}}{2}=\frac{4\pm 2}{2}=\begin{cases} x=3 \\ x=1 \end{cases}$$

[3] In the number sets **Q** and **R** an order relation can be defined.

As we have previously pointed out, we only present a sample of all the possible definitions. Other characterizations of absolute value can be based, for example, on the notion of vector norm (Mollin, 1998, p.47). What is essential in this paper is not the number or the particular types of definitions used, but the fact that several definitions for the same mathematical notion coexist, that each of them involves a network of related objects and that all these definitions interact. All of this, together, lead to a high semiotic complexity.

Short analysis of calculation solution techniques of $|x - 2| = 1$

The aforementioned definitions are mathematically equivalent, but their use conditions the mathematical activity. In particular, their use conditions the resolution techniques of the equation $|x - 2| = 1$, as we have shown. For this reason, we can affirm that the definitions are not equivalent from the epistemic point of view: they do not involve the same mathematical objects in the resolution of a problem and, therefore, they condition the operative and discursive practices in relation to the AVN.

The proofs we have carried out involve applying some general techniques for solving linear equations with an absolute value (namely, equations of the type $| x - k | = m; k, m \in \mathbf{R}$) to a particular case. In the epistemic analysis of a mathematical object, it is necessary to answer a basic question: What is the applicability field for the techniques? For example: Do they allow equations of the form $| r.x - k | = | s.x - t |$ $(r, k, s, t \in \mathbf{R})$ to be solved? Is it necessary to make substantial adaptations or simple adaptations to be able to solve these problems? What instruments of evaluation do we have to judge if these technical adaptations are substantial or simple?

Nevertheless, if we restrict the analysis to solving linear equations with absolute value of the form $| x - k | = m$ $(k, m \in \mathbf{R})$, the shown techniques must solely be analysed in terms of their effectiveness, students' cognitive cost depending on their mathematical background and the necessary resources for teaching and learning. The analysis of the educational effectiveness of the techniques involved, particularly, the determination of the type of relations that exist between these techniques. Next we shall show the onto-semiotic complexity of the AVN, that is deduced from the diversity of contexts of use, from the definitions associated with them and the operative and discursive practices that these definitions condition. Then, backed by the theoretical notion of holistic meaning, we shall organise the partial meanings of absolute value, whilst showing the relations that are established between them.

ONTO-SEMIOTIC COMPLEXITY OF ABSOLUTE VALUE

Structure of definitions, objects and meanings associated with the notion of absolute value

The professional mathematician identifies the same formal structure in the variety of objects and (operative and discursive) practices; a structure that he/she considers to be "the mathematical object". This formal structure represents the implicit reference in the resolution of types of problems associated with the variety of systems of practices and objects emerging in the different contexts of use.

As suggested by Ullmann (1962, p. 76), researchers should first gather an appropriate sample of contexts and approach them later with an open mind, thus allowing the meaning or meanings to emerge from these contexts. Once this phase has been concluded, we can safely enter into the "referential" phase and try to formulate the meaning or meanings identified in this way. Our meaning begins by being pragmatic, relative to the context, but there are typical uses that allow us to guide mathematical teaching and learning processes. These types are objectified by language and constitute the referents of the institutional lexicon.

Figure 1 schematically shows the diversity of objects associated with the AVN. Each definition represents an object emerging from a system of practices in a given context of use. No definition may be privileged a *priori*. Each "emergent object - system of practices" binomial determines a *partial meaning* of the AVN. The partial meanings are then a coherent form for structuring the different contexts of use, the mathematical practices relating to them and the objects emerging from such practices; so forming a network or *local epistemic configuration* (associated with a specific context of use).

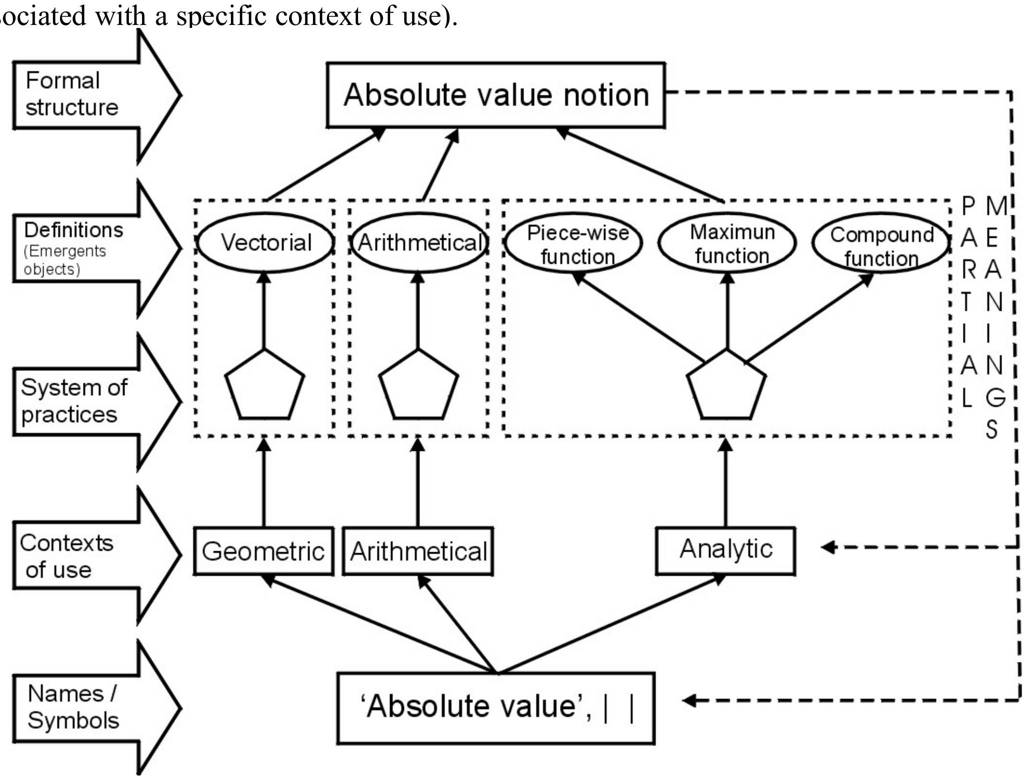

Figure 1. Structure for the objects and meanings associated with the absolute value.

In Figure 1 we do not represent a chronological order in the process of conceptualising AVN, but the structure of the different aspects that intervene in this process. Names and symbols adopt a specific sense in each context of use and the formal structure provides a coherent reference for the different partial meanings.

In the ontosemiotic approach to mathematics cognition *meaning* is the content of any semiotic function, that is to say, the consequent in the correspondence we establish between two mathematical objects in a communicative act. The antecedent is usually, but not restricted to a linguistic object. The answer proposed by the ontosemiotic approach to the question, *what does "absolute value" means?* is diverse, according to the contexts, the language games, people or institutions involved. Meaning might be one definition, the system of practices associated to a given context, or even a student's personal interpretation (his/her personal meaning of absolute value). Meaning of absolute value (derivative, function, etc.) is also dynamic, since it "grows" when the problems and practices are generalised.

The definitions described represent the objective elements of these systems of practices and, therefore, their relational description will determine an explicit partial meaning for these systems. Below we introduce the idea of *holistic meaning* to represent a relational structuring of the associated mathematical partial meanings for the AVN.

Holistic meaning of the notion of absolute value

From the strictly *formal and official* viewpoint (Brown, 1998), it is accepted that the definition of a mathematical object constitutes its meaning. The description of the system of partial meanings associated with a notion is obtained from the statement and demonstration of a characterisation theorem: usually this privileges one definition and requires the justification of the equivalence of the other definitions.

Based on empirical data, Leikin and Winicki-Landman (2000) emphasize different strategies used by teachers to analyse equivalent definitions of a mathematical notion. The application of these strategies is not restricted to the epistemological dimension. "The teachers usually discussed both pedagogical and mathematical characteristics of the definitions, and often used didactical considerations in order to explain their mathematical preferences" (Leikin y Winicki-Landman, 2000, p.27). In fact, the empirical data provided by Leikin & Winicki-Landman allow to be stated that the equivalence of mathematical definitions cannot be assessed just from the epistemological viewpoint, it is necessary take into account the *cognitive* (What strategies for action generate each one of the definitions?), *instructional* (What definition is the most suitable within a given project for teaching?) and *didactic* (What relationship is established between the personal meaning learnt and the institutional meaning intended?) dimensions. The *holistic meaning* comes from the coordination of the partial meanings associated with AVN and the tensions, filiations and contradictions that are established between them and is relative to the "ideal mathematical-didactics institution", where absolute value has a unique "global" reality.

Hence, the justification of the mathematical equivalence of the definitions of absolute value does not represent a suitable instrument for structuring the partial meaning associated to that notion from the teaching and learning point of view. The study of the partial meanings of the absolute value (with their associated definitions) and the application for solving the linear equations show the need for a flexible transit between the different partial meanings. Tall (1991) defines the *reflective abstraction* as a mechanism whereby the subject encapsulates a process in a concept. It is precisely this encapsulation that enables the professional mathematicians to use, in a flexible way, a symbol that represents a process and a concept at the same time. That is to say, reflective abstraction is the basis of so-called "flexible mathematical thought", that allows passing through both states of a mathematical object in a way that is not abrupt: *operational dynamic* (when the object is used as an instrument, that is to say, understood as a process) and *structural static* (when the object is related to other objects in a theory, that is to say, understood as a concept).

The definition introduced by Tall is not operative (Williams, 1999): how can we determine whether a subject performs a mathematical activity using the dual character process-concept of a mathematical object in a flexible way? What observable actions, gestures, arguments constitute evidence of the fact that a subject operates by means of a flexible mathematical thought in a process of mathematical study? Is flexible mathematical thought a *metaphor* (Barrow, 1997) for determining an optimal behaviour with regard to a system of operational and discursive practices linked to a mathematical object?

Our view is that flexible mathematical thought represents an action carried out by a subject that allows the routine transit between different associated partial meanings of a mathematical object (recognizing the limitations of each partial meaning). The subject also establishes, by means of flexible mathematical thought, links between these partial meanings and one or several mathematical fields. Those links determine an effective control of the activity and enable the subject to take mathematical responsibility for the results that he/she produces. The holistic meaning is the result of the mathematician's activities in the coordination process for the partial meanings and, also, the tensions, connections and contradictions that he/she establishes between the partial meanings. The mathematician controls this coordination process using flexible mathematical thought.

COGNITIVE EFFECTIVENESS OF THE ARITHMETIC AND "PIECE-WISE FUNCTION" PARTIAL MEANINGS OF THE ABSOLUTE VALUE

In our theoretical framework, the epistemic dimension for a didactic problem is always institutional and might refer to the teaching process (*implemented meaning*, which is the immediate reference for students' learning), the planning of that teaching process (*intended* *meaning*), or a global reference wider than the planned teaching (*holistic meaning*). The cognitive dimension in this framework refers to the students' personal objects and meanings. This institutional–personal facet serve to explain the students' learning (as well as its limitations and conflicts) in function of the relationships between each student's personal meaning and the implemented institutional meanings.

In this section we explore the cognitive dimension in a teaching process of AVN from the students' responses to a questionnaire. This was part of a wider investigation that included revision of syllabuses, preparation of a textbook and teaching materials, observation of teaching sessions and interviews with students. The analysis made in previous sections of AVN holistic meaning served in that research as the reference in the design of teaching and evaluation of learning.

As mentioned earlier, from the viewpoint of mathematics education, a fundamental question consists of determining the *didactic effectiveness* of a mathematical process for problem-solving. In section "Definitions of absolute value" we showed how the techniques associated to the definitions of absolute value allowed the "linear equations with absolute value" type of problems to be solved. This fact did not mean an epistemic equivalence, since the techniques do not involve the same mathematical objects. In this section we aim to analyse the *cognitive* dimension (effectiveness and cost in the use of the techniques by individuals) of the problem-solving techniques associated with the "arithmetical" definitions and "piece-wise function". To do so, we use an experimental study with a group of 55 students (secondary trainee teachers) solving a set of elemental exercises that require the AVN (table 1). The *implicative* and *hierarchical* statistical method (Gras, 1996) was used to analyse the responses given by the students.

Table 1. Questionnaire.

1. Complete, when possible, the following equalities:			
$\|-2\| =$	$\|2\| =$	$\|0\| =$	$\|\sqrt{-2}\| =$
$\|\sqrt{2}\| =$	$\|-\sqrt{2}\| =$	$\|2-\sqrt{2}\| =$	$\|\sqrt{2}-2\| =$
2. Replace the dots, when possible, in the following expressions to make them correct:			
$\|\ldots-2\| = 1;$	$\|\ldots+2\| = 1;$	$\|\ldots-2\| = 0;$	$\|(\ldots)^2-4\| = 0;$
$\|(\ldots)^2+4\| = 0;$	$\|(\ldots)^2-1\| = 1;$		$\|(\ldots)^2-3\| = 1$
3. Represent the function $f(x) = \|x+1\|$ in a graphical way.			
4. Let a be a real number. Complete, when possible, the following equalities:			
$\|-a\| =$	$\|a\| =$	$\|a-2\| =$	
$\|-a-2\| =$	$\|2-a\| =$	$\|a+2\| =$	

Institutional implemented meaning in the study process

The main goal of the empirical study was to reveal the agreements and mismatches between the institutional intended meaning and the achieved personal meaning, which were determined after an in-depth personal interview of the teacher carried out by a researcher. It was complemented with content analysis of syllabuses, the textbook and materials used in the teaching.

The intended meaning was reduced to the relationship between the "arithmetic", "piece-wise function" and "metric" partial meanings. In the institution, the metric partial meaning also plays a merely descriptive role of the action "remove the minus sign from a negative number". Therefore, the intended meaning is structured for achieving the following objectives: 1) introduce the absolute value notion using the arithmetic partial meaning; 2) develop this partial meaning for the formulation of the piece-wise function partial meaning; 3) accept that the piece-wise function partial meaning is a formalization of the arithmetic partial meaning that is most effective in the problem-solving process. Nevertheless, the cognitive and mathematical students' restrictions and the didactical teachers' restrictions limit the implemented meaning to a collection of manipulations with particular numbers and the evaluated meaning to routine activities for obtaining the absolute value of real numbers ($|-2|$, $|2|$, $|0|$, $\left|2-\sqrt{2}\right|$, $\left|\frac{1}{2}-\frac{1}{\pi}\right|$, etc.), as well as to the graphical representation of the absolute value of linear functions by means of the "point to point technique".

Effectiveness in problem-solving

Generically, we affirm that a person understands the AVN if he/she is capable of distinguishing its different associated partial meanings, structuring the said partial meanings in a complex and coherent set and meeting the operative and discursive practices needed in relation to the AVN in the different contexts of use. Bills and Tall (1998) establish that a definition is operative for a subject when he/she can use this definition in a pertinent way in a logical-deductive demonstration process. From our viewpoint, the operating capacity of a mathematical object in a study process needs a balance between the operating role and the theoretical or discursive function. In a global education project, the analysis of the operating capacity should also take into account, in addition to the cognitive dimension, the epistemic and instructional dimensions.

The notion of "didactic effectiveness" introduced earlier takes into account these three dimensions and incorporates the operative and discursive functions of mathematical activity. Then, in relation to the arithmetic partial meaning and piece-wise function partial meanings we ask ourselves: Is it possible to discriminate those partial meanings by their effectiveness and cost in the use of the techniques by the subjects?

The only means for distinguishing the meaning attributed by an individual to an object is through a situation or a set of problems that may be solved by using different partial meanings capable of generating pertinent and useful actions.

The experimental work performed has allowed us to classify the students according to the partial meaning of absolute value associated with the operative and discursive practices in relation to the problems proposed (that determines a certain level of effectiveness). So as to be able to classify the students, it is necessary to interrelate a collection of tasks and determine (with a level of approximation) the tasks that allow the performance of other tasks to be assured.

Analysis of a questionnaire

The main purpose of the teaching experiment is to empirically support or not the thesis according to which the partial meanings “arithmetical” and “piece-wise function” associated with the AVN are quite similar. The a priori analysis provides criteria to select variables to carry out the implicative and hierarchical study (Gras, 1996).

A priori analysis

The section “Onto-semiotic complexity of absolute value” provides the a priori analysis of the AVN institutional meanings. The contrast between this a priori analysis and the students’ effective realizations is described in terms of suitability of a study process, that is, fitting of intended, effectively taught and achieved meaning. Some suitability criteria, which can be synthesized taking into account the epistemic, cognitive and instructional dimensions, will be introduced below.

The evaluation of epistemic suitability consists assessing the agreements and disagreements between the reference institutional and effectively taught meanings. Cognitive suitability assesses whether the proposal took into account the students’ cognitive restrictions and whether the material and temporal resources were enough to overcome gaps between initial personal meanings and the institutional meanings we wanted to teach. Finally, instructional suitability refers to teaching mechanism, to identify semiotic conflicts and solve them by negotiation of meanings.

The validity of the questionnaire given in Table 1 should be assessed from the epistemic, cognitive and instructional dimensions. For example, in question 1 we expect the answer “it is a complex number”, “non sense”, “you cannot take out square root of a negative number”, when asking students to determine $|\sqrt{-2}|$. The institutional restrictions determine the absolute value function to be defined in the real numbers set (epistemic dimension); a problem have one, none or several solutions (cognitive dimension); and, finally, the whole study process has been focused on the absolute value of real numbers (instructional dimension).

Implicative and hierarchical study

In the implicative analysis (Gras, 1996), the aim is to find whether, in the sample, the fact of having answered a question correctly statistically implies the response to another question. In particular, it is admissible to expect that any individual who is capable of performing a task that is more complex than another (and that generalises it in a certain way), then he/she will also be capable of performing the second one. However, this is not always so: the notion of "difficulty" is associated to the naturalized practices in a particular institution and the personal meaning that is not always specified; therefore, the resolution expectation can be "contradicted" in a particular sample (by an ample group of subjects). That is why, in many circumstances, it is necessary to compare certain hypotheses for implementing a hierarchy for performing tasks.

A wider question that may be posed is whether the fact of having answered a set of questions correctly implies (in a preferential manner) the right answer in another set of questions. The hierarchical analysis, defined on the basis of the cohesions between classes, allows the implicative relationships between the kinds of questions to be described in a more "dynamic" way and, therefore, constitutes a response to the question posed.

Results

Implicative and hierarchical analysis provide the following results:

- The arithmetical partial meaning of absolute value is understood as a rule that operates on the "numbers", that is to say, numbers "in decimal format".
- The students who have a "good" behaviour in the course mostly operate the absolute value "symbolically" ($|\sqrt{2}-2|=2-\sqrt{2}$) and understand the f(x) = $|x + 1|$ function analytically and graphically.
- The group of students can be structured in two subgroups, each of them with stable answers in relation to the tasks they carry out (and the meanings of absolute value applied).
- The students performing symbolically the tasks and capable of systematically and effectively applying the piece-wise function partial meaning form one group; the other being characterised by application of the arithmetical partial meaning.
- The piece-wise function partial meaning is essential for the effective accomplishment of tasks related to the AVN. More specifically, we expect an implication from tasks requiring analytical partial meaning to tasks needing the arithmetic partial meaning.
- The analysis of the questionnaire, observation of teaching sessions and interviews with the teacher and students, showed that the teaching carried out before applying the questionnaire, might be suitable from the cognitive point of view since the time and material resources allowed student to solve the tasks proposed during this study process. However, the epistemic and

instructional suitability might be low. On one hand, the taught institutional meaning did not allow most students the formalization of AVN in terms of the "piece-wise function" partial meaning (intended institutional meaning). On the other hand, the tasks proposed in the study process did not allow the identification of conflicts of meaning related to AVN (as it was shown in the students' answers to the questionnaire).

Brief discussion

The questionnaire has related two partial meanings of the AVN: the arithmetical and the analytical "piece-wise function". The emergence of one or another partial meaning conditions the performance of the tasks proposed in the questionnaire (Table 1), at least for the sample studied. In fact, despite the fact that a priori (mathematically) the first two exercises and the last one may be solved "by the systematic application of the rule associated with the arithmetic partial meaning", what is true is that the students who only have this partial meaning encounter serious problems for solving the tasks. Just referring to the first two exercises and the last one, it may be affirmed that the analytical partial meaning is preferred due to its greater heuristic power, despite the fact that "in principle" both partial meanings allow the "complete" solution of the exercises.

The students with the "piece-wise function" partial meaning are capable of using the absolute value symbolically and of systematizing the use of the arithmetical partial meaning. However, the students who only have the arithmetical partial meaning associate the AVN with the algorithmic action:

$$[a \in \mathbf{R} \rightarrow \text{ decimal expression of } a \rightarrow \text{ sign}(a)] \rightarrow |a|$$

The arithmetical partial meaning is transformed in the statement: "the absolute value is a rule that removes the minus sign from the negative numbers"; i.e., a rule for annotation: the ostensives "3" and "|–3|" represent the same object, they constitute a case of synonymy. The students perform symbolic manipulations with "hidden" positive numbers. If the goal of teaching is that the students learn this partial meaning, the mathematical work is centred on the technical use of the absolute value. The "good" student is the one who accepts the rules imposed by the teacher, who accepts playing the game with the symbols and acquires great mastery in this game.

Hence, it is necessary to make a didactic contract with severe restrictions (which alters the original mathematical interest of the object, with it being relegated to a simple game of symbols). The situation is similar to when a child "sings" the numbers or recites the alphabet, like a learned sequence of sounds, ordered for him/her by an arbitrary criterion; the only reason by which the child sings the numbers "in an ordered way" or recites the alphabet correctly is affective: he/she receives the adult's praise and, possibly, a material prize. Then, it is clear that the same old story taught in an different order would be recited with the same rash enthusiasm.

MACRO AND MICRO DIDACTIC IMPLICATIONS

Cognitive difficulties (Chiarugi, Fracassina & Furinghetti, 1990) and the incapacity of educational institutions to draw up a pertinent curriculum to introduce and develop the AVN (Perrin-Glorian, 1995; Gagatsis and Thomaidis, 1994) has led to merely technical teaching based on the arithmetical partial meaning (as a rule that "removes the minus sign"). The arithmetical partial meaning of the AVN proves to be a didactic obstacle that restricts, in many cases, the personal meaning to a mere game of symbols. This obstacle is shown in different ways; for example: $|a| = a$ and $|-a| = a$, for any $a \in \mathbf{R}$; $|\sqrt{2}-2| = \sqrt{2}+2$, etc.

Macrodidactic implications

As Winicki-Landman & Leikin (2000, p.17) affirm "one of the more important questions in mathematics education is: 'What is the best way to introduce a new mathematical concept to a learner?" Teaching a mathematical notion by means of an associated partial meaning, it is necessary to ensure its representativeness in relation to the institutional referential meaning. The introduction of the absolute value by means of the arithmetic partial meaning is not representative: any analytical partial meaning cannot be dealt with guarantees (the theory of functions is beyond the students' knowledge); the vectorial partial meaning can only be described in natural language (not formalized); and, finally, the geometric partial meaning is understood as a simple rule "to delete the minus sign". Hence, the introduction of the absolute value in the arithmetical context represents an unfortunate decision in modern-day school institutions: it means the inclusion in the curriculum of the notion of "absolute value" for merely cultural reasons. However, the curricular structure is not ready at the present to properly cope with the study of this notion in an exclusively arithmetical context. It would be advisable to "temporarily" remove the notion. This would be temporary, either until a pertinent didactic transposition, or until the students start to study the theory of functions, central in relation to the notion of absolute value (Arcidiacono, 1983; Horak, 1994).

Microdidactic implications

For the AVN, the partial meaning "piece-wise function", using the graphic representation of the function in the Cartesian plane, pertaining to the theory of functions. Hence, it is necessary to establish a didactic engineering for developing the AVN in the setting of theory of functions. This engineering will have to articulate the epistemological analysis with the methodological and time restrictions within each specific institution. In relation to the AVN, the objective consists of establishing a system of practices that will make the explicit interaction of the arithmetical partial meaning with the rest of the partial meanings possible and, most particularly, with the analytical partial meaning.

SYNTHESIS AND THEORETICAL IMPLICATIONS

In this paper we have contributed a new perspective about two theoretical problems of interest for the epistemology of mathematics and mathematics education:

1. Global and partial meanings of mathematical objects. Even when a mathematical object (AVN, tangent, real number, function, etc.) can be characterized by diverse definitions that are formally equivalent, some of these definitions might be not equivalent from the epistemic, cognitive or instructional viewpoints. Each definition is linked to a configuration or network of other objects and relationships, which efficiently allow the solving of a certain class of problems. Different definitions are not equally effective, that is, the possibilities of generalizing their associate techniques and their ontosemiotic complexity involved in their study vary.

   In consequence, the didactic analysis of a mathematical object should characterize the diverse configurations composing its global or holistic meaning, in order to adopt decisions related to representative and effective selection of meanings in each specific educational circumstance.

2. Coherent articulation between the pragmatic and realist visions of mathematics. Building and communicating the meanings of mathematical objects require, on one hand, recognizing an institutional and contextual relativity for the same and, on the other hand, accepting the existence of the realistic-referential viewpoint for these objects usually assumed by professional mathematician. Articulating these two visions supposes the recognition of an intra-mathematics context of use, where the description of the formal structure (common to other contexts of use) and the foundation of mathematics as a body of knowledge is carried out.

These two theoretical contributions have implications about curricular design questions, as well as on the teaching and learning mathematics. The notion of holistic meaning of a mathematical notion provides an instrument for controlling and assessing the systems of practices implemented and an observable response for the analysis of personal meanings. More precisely speaking:

- The notion of holistic meaning (network of partial meanings) represents the structuring of the knowledge targeted and may be used to determine the degree of representation of a system of practices implemented in relation to the institutional meaning intended.

- The notions of partial and holistic meanings provide a response to the questions: What is a mathematical notion? What is understanding this notion?; in particular, What is the AVN? What does understanding the AVN mean?

REFERENCES

Arcidiacono, M. J. (1983). A visual approach to absolute value. *The Mathematics Teacher*, 76 (3), 197?201.

Barrow, R. (1997). Language: definition and metaphor. *Studies in Philosophy and Education*, 16, 113?124.

Bills, L., and Tall, D.(1998). Operable definitions in advanced mathematics: the case of the least upper bound. In A. Olivier and K. Newstead (eds.), *Proceedings of PME 22* (Stellenbosch, South Africa), Vol. 2, 104?111.

Brown, J. R. (1998). What is a definition?. *Foundations of Science*, 1, 111?132.

Chiarugi, I., Fracassina, G., and Furinghetti, F. (1990). Learning difficulties behind the notion of absolute value. *Proceedings of PME 14* (México), Vol. 3, 231?238.

De Villiers, M.(1998). To teach definitions in geometry or teach to define?. In A. Olivier and K. Newstead (eds.), *Proceedings of PME 22* (Stellenbosch, South Africa), Vol. 2, 248?255.

Gagatsis, A. (2003). A multidimensional approach to understanding and learning mathematics. In Gagatsis and Papastavridis (eds), *Proceedings of the 3rd International Mediterranean Conference on Mathematics Education*. (pp. 53–72). Athènes: Hellas.

Gagatsis, A., and Thomaidis, I. (1994). Un étude multidimensionnelle du concept de valeur absolue. In M. Artigue, R. Gras, C. Laborde and P. Tavignot (eds.), *Vingt ans de didactique de mathematiques en France*. (pp. 343?348). Grenoble: La Pensée Sauvage.

Godino, J. D. (2002). Un enfoque ontológico y semiótico de la cognición matemática. *Recherche en Didactique des Mathématiques*, 22(2/3), 237?284.

Godino, J. D. and Batanero, C. (1998). Clarifying the meaning of mathematical objects as a priority area of research in mathematics education. In A. Sierpinska and J. Kilpatrick (Eds.), *Mathematics Education as a Research Domain: A Search for Identity* (pp. 177-195). Dordrecht: Kluwer, A. P.

Godino, J. D., Batanero, C., and Roa, R. (2005). An onto-semiotic analysis of combinatorial problems and the solving processes by university students. *Educational Studies in Mathematics*, 60 (1), 3 - 36.

Gras, R. (1996). *L'implication statistique*. Grenoble, FRA: La Pensée Sauvage.

Leithold, L. (1968). *The calculus with analytic geometry*. New York: Harper & Row.

Horak, V. M. (1994). Investigating absolute-value equations with the graphing calculator. *The Mathematics Teacher*, 87(1), 9?11.

Leikin, R., and Winicki-Landman, G. (2000). On equivalent and non-equivalent definitions: part 2. *For the Learning of Mathematics*, 20(2), 24?29.

Linchevsky, L., Vinner, S., and Karsenty, R. (1992). To be or not to be minimal? Student teachers' views about definitions in geometry. *Proceedings of PME 16* (Durham, NH), Vol. 2, 48?55.

Mariotti, M. A., and Fischbein, E. (1997). Defining in classroom activities. *Educational Studies in Mathematic*s, 34, 219?248.

Mollin, R. A. (1998). *Fundamental number theory with applications*. Boca Raton, Florida: CRC Press.

Perrin-Glorian, M-J. (1995). The absolute value in secondary school. A case study of "Institutionalisation" process. *Proceedings of PME 19* (Recife, Brazil), Vol. 2, 74?81.

Tall, D. (1991). The psychology of advanced mathematical thinking. In D. Tall (ed.), *Advanced Mathematical Thinking* (pp. 3?21). Dordrecht, HOL: Kluwer.

Ullman, S. (1962). *Semántica. Introducción a la ciencia del significado*. Aguilar, Madrid, 1978.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (ed.), *Advanced Mathematical Thinking* (pp. 65?81). Dordrecht: Kluwer.

Williams, R. L. (1999). Operational definitions and assessment of higher-order cognitive constructs. *Educational Psychology Review*, 11(4), 411?427.

Winicki-Landman, G., and Leikin, R. (2000). On equivalent and non-equivalent definitions: part 1. For the Learning *of Mathematics*, 20(1), 17?21.

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