The Influence of Teaching on Student Learning: The Notion of Piecewise Function

İbrahim Bayazit
Erciyes University

This paper examines the influence of classroom teaching on student understanding of the piecewise function. The participants were two experienced mathematics teachers and their 9th grade students. Using a theoretical standpoint that emerged from an analysis of APOS theory, the paper illustrates that the teachers differ remarkably in their approaches to the essence of the piecewise function and this, in turn, affects greatly their students' understanding of this notion. Action-oriented teaching, which is distinguished by the communication of rules, procedures and factual knowledge, confines students' understanding to an action conception of piecewise function. Process-oriented teaching, which priorities the concept and illustrates it across the representations, promotes students' understanding towards a process conception of function.

Keywords: action-oriented teaching, process-oriented teaching, student learning, piecewise function, action conception of piecewise function, process conception of piecewise function

The impact of teaching practices on student learning has been a focus of research for several years (Brophy & Good, 1986; Helmke, Schneider, & Weiner, 1986; Weinert, Schrader, & Helmke, 1989; Cobb, McClain, & Whitenack, 1997). Prompting this interest is the belief that teachers play an active and direct role in students' knowledge construction. Conventionally, studies that examine the influence of teaching on student learning are called 'process-product' research (Brophy & Good, 1986; Pirie & Kieren, 1992; Askew, Brown, Rhodes, William, & Johnson, 1996). These studies differ however in their methodological approaches and their particular focus on the social, psychological, and pedagogical aspects of teacher's classroom practices and relate them to student learning; thus they could be considered in two groups, namely 'simple process-product research' and 'qualitative process-product research.' Those in the former group (Good & Grouws, 1977; Tobin & Capie, 1982) focused, mainly, upon directly observable teaching inputs and related them to the students' achievements as measured by means of standard tests. Development in this field has been well documented by Brophy and Good (1986) who, after reviewing the literature, reported several teaching inputs – such as having good relation with the students and the amount of time spent for instructive purposes – which are positively associated with the students' high achievements in mathematics.

Qualitative process-product research employed in-depth qualitative inquiry to gain better understanding of social, psychological and pedagogical aspects of teaching, learning, and the interactions between the two (e.g., Pirie & Kieren, 1992; Askew et al., 1996; Fennema et al., 1996; Cobb et al., 1997). Pirie and Kieren (1992) conceptualized teaching as the continuing act of creating learning opportunities, and they considered learning as an individual's mental processing of the knowledge offered by those environments. Conducting in-depth analysis of teacher-student exchanges the authors indicated that constructivist teaching approach helped students to develop conceptual knowledge of the fractions. The distinguishing aspects of constructivist teaching approach include, for instance, presenting the concept in a manner that allows students to develop images of fractions through experiencing concrete materials (e.g., folding a paper into half), and utilizing students' primitive (tacit) knowledge of fraction to support their formal growth in the concept. Cobb et al. (1997) examined the influence of classroom discourse on students' understanding of the arithmetical concepts. They identified two crucial features of classroom discourses: 'reflective discourse' and 'collective reflection.' Reflective discourse is characterized by "repeated shifts such that what the teacher and students do in action subsequently becomes an explicit object of discussion" (p. 258), whilst collective reflection is distinguished by the "communal effort to make what was done before in action an object of reflection" (p. 258). The authors suggest that these aspects of classroom discourse prompted students' development of the arithmetical concept from an action-process conception (e.g., applying counting strategies to solve simple arithmetic problems) to an object conception (e.g., using mental strategies that include the conceptual coordination of units of ten and one in solving arithmetic problems). In this paper, the notions of *reflective discourse* and *collective reflection* are used to differentiate the cognitive focus of the teachers' classroom practices.

The present study fits well into the tradition of 'qualitative process-product' research. It examines two experienced Turkish teachers' instruction of piecewise functions and relates it to their students' learning of this notion. It contributes to the literature by identifying two teaching orientations: process-oriented teaching and action-oriented teaching (Bayazit, 2006), and indicates that these teaching orientations would produce qualitatively different learning outcomes: process-oriented teaching could promote students' understanding toward a process conception of piecewise function whilst action oriented teaching could confine their understanding to an action conception of piecewise function.

Developing a Theoretical Framework

The Turkish mathematics curriculum introduces piecewise functions at the 9th grade level, and illustrates them further at the 11th grade level through particular types of functions including absolute value functions, integer functions, and sign functions. A piecewise function, defined by more than one rule on the sub-domains, does not violate the definition of the function (concept definition). Nevertheless, most students think that a function should be described with a single rule over the whole domain (Sfard, 1992). Involvement of more than one rule in a situation can result in the misconception that the situation represents two or more functions, not just one (Markovits, Eylon, & Brukheimer, 1986). A graph made of branches or discrete points could denote a piecewise function on a split domain; nevertheless students usually reject such graphs because they possess a misconception that a graph of function should be a continuous line or curve (Vinner, 1983; Breidenbach, Dubinsky, Hawks, & Nichols, 1992). It is suggested that students would overcome such difficulties and misconceptions as they develop a process conception of function (Dubinsky & Harel, 1992).

In this paper I refer to the notions of *action* and *process* conceptions of function – the first two stages of APOS theory – to examine the teachers' instructions of the piecewise function and their students' resulting understanding of this notion. Inspired by Piaget's idea of reflective abstraction Dubinsky (1991) introduced APOS theory in an attempt to illustrate mental processing of mathematical notions and what can be done to help individuals in their learning. The theory has four components including action, process, object and schema. It has been used as a theoretical framework by many scholars in different type of studies (see for instance, Breidenbach et al., 1992; Cottrill et al., 1996; Bayazit, 2006). The theory of APOS has both advocates and opponents. Advocates of the theory believe that it is very useful in attempting to understand students' learning of a broad range of mathematical topics including algebra and discrete mathematics (Eisenberg, 1991; Cottrill et al., 1996) whilst the opponents criticizes the universal applicability of APOS and claim that it lacks an ability to explain construction of geometrical concepts (Tall, 1999). In the following, I illustrate the stages of APOS theory with specific reference to the function concept.

An action conception of a mathematical idea refers to repeatable mental or physical manipulations that transform objects (e.g., numbers, sets) into new ones (Cottrill et al., 1996). Understanding reflecting such a conception suggests an ability to complete a transformation by performing all appropriate operational steps in a sequence. Dubinsky and Harel (1992) indicated that such a conception involves the ability to substitute a number into an expression and calculate its image. However, understanding restricted to actions suggests that learners would compose two algebraic functions by replacing each occurrence of the variable in one expression by the other expression and simplifying (Breidenbach et al., 1992). It is conjectured that an action conception of function enables one to perform mechanical manipulations with the algebraic piecewise functions. For instance, those who possess an action conception would compose two piecewise functions at a point when the functions are defined by the algebraic expressions. They would obtain the images of inputs by inserting the elements into the expression(s) and making step by step calculations.

A process conception of function is considered at a higher level in that the possessor is able to internalize actions and talk about a function process in terms of input and output without necessarily performing all the operations of a function in a step-by-step manner (Breidenbach et al., 1992). A process can be manipulated in various ways; it can be reversed or combined with other processes (Dubinsky & Harel, 1992; Cottrill et al., 1996). The possession of a process conception allows students to recognize a single function process represented by more than one rule on the sub-domains and interpret the process in light of concept definition without losing the sight of univalence condition. Dubinsky and Harel (1992) assert that the possession of a process conception is critical to overcome the continuity restriction, which concerns a misconception that a graph of function should be a continuous line or curve – the ability to interpret a function process in a graph made of discrete points is indicator of a strong process conception.

Even though it is not at the heart of discussion within this paper, it is worth considering the notions of *object conception of function* and *schema*. Constant reflection upon a process may lead to its eventual encapsulation as an object (Cottrill et al., 1996). Within the function context, the possession of object conception entails the ability to use a function in further processes, and with this understanding a function may be used in the process of derivative and integral. From the graphical perspective, an object conception of function enables one to manipulate a graph of function (e.g., shifting the graph of $f(x)=2x^2$ three units through the *y*-axis in the negative direction to obtain the graph of $g(x)=2x^2-3$) without dealing with the graph point by point. Finally, a schema refers to a collection of actions, process and objects that an individual possess about a mathematical notion (Dubinsky, 1991). It is some sort of mental framework that an individual bears upon a problem situation involving that concept. One's schema of functions may include action, process and object conceptions of functions, associated rules and procedures, mental images, analogies, and prototypical examples related to the concept of function.

Although the notions of action and process are introduced to interpret the quality of students' understanding of algebraic concepts, I utilized these notions to identify the cognitive focus and the key aspects of the teachers' classroom practices. Arising from the above discussions (Breidenbach et al., 1992; Cottrill et al., 1996) this paper illustrates two different teaching approaches: action-oriented teaching and process-oriented teaching (Bayazit, 2006). Action-oriented teaching is distinguished by the teacher's instructional acts which emphasize step-by-step manipulation of algorithmic procedures and engage students with the visual properties of algebraic piecewise functions. The essence of process-oriented teaching is that the teacher prioritizes the concept and illustrates it across the representations. Process-oriented teaching uses the *concept definition* (Vinner, 1983) as a cognitive tool and provides concept-driven, clear, and explicit verbal explanations to facilitate students' accession to the idea of piecewise function in the algebraic and graphical context. The notions of action-oriented and process-oriented teaching are not static but dynamic constructs; thus I shall point out the aspects of these teaching orientations as we present the data in the coming sections.

Research Method and Data Analysis

This research study employed a qualitative case study (Merriam, 1988) to interpret the teachers' classroom practices and their possible impacts on students' learning as closely as possible. The participants were two experienced teachers (Ahmet[1] had 25 years of teaching experience and Burak had 24 years of teaching experience) and their 9th grade students (age 15). A purposeful sampling strategy (Merriam, 1988) was employed to involve teachers who had different conceptions about teaching functions, but also to control students' initial levels, their socio-economic backgrounds, and other school/teacher-related factors including, for instance, instructional facilities provided by the case schools and the teachers' formal qualifications in mathematics education. Twelve teachers within four different schools were initially visited to gain, through informal interviews, ideas about their teaching approaches to functions. Most revealed similar views that favored mechanical manipulations with the algebraic functions. Having considered the research goal and the practical issues on the ground two teachers from two different schools were chosen for the main study (Ahmet from School A and Burak from School B). During the informal interview Ahmet and Burak reflected different views about teaching functions. Ahmet stated his belief about the effectiveness of constructivist teaching approach. He emphasized that he liked teaching the essence of the function concept and described the essence of the concept as the *concept definition*. In contrast, Burak revealed a kind of behaviorist approach towards teaching the

[1] Teachers' and students' names are pseudonyms, and the classes are identified by the initial of teachers' names – Ahmet's Class: Class A, Burak's Class: Class B.

function concept. He stated his belief that analogy and the use of real life examples would make the function concept comprehensible to the students. Nevertheless, his comments implied that Burak appreciated the dispense of rules, procedures and the factual knowledge to promote his students' competence in manipulating algebraic functions.

Data was obtained from multiple sources. Each teacher was observed for 14 lessons (each lasting 45 minutes) including the sessions they taught piecewise functions. Lessons were tape-recorded and annotated field notes were taken to record social, psychological and pedagogical aspects of the teaching practices and the visual attributes that the audiotape could not detect. Data associated with student learning was collected through pre- and post-tests which included an opportunity for students to express their actual thinking. Semi-structured interviews with three students from each class were carried out after each test. The interviewees were selected on the basis of their achievements in the tests and the teacher recommendations. Accordingly, three students (one from low achievers, one from medium achievers, and one from high achievers) from each class were chosen for the interviews.

More than one method and various strategies were used to analyze the data collected. Qualitative method, discourse and content analysis (Philips & Hardy, 2002), were used to analyze the transcripts of the lessons and the student interviews. In particular, the method of discourse analysis aimed not to interpret a specific instructional act in its own context or in the context of a single lesson but to construe the act in the whole of teachers' instructions on functions unit. Lessons were fully transcribed and considered line by line while annotated field notes were used as supplementary sources. The first phase of analysis included assigning initial codes (brief descriptions) to the meaning inferred from the texts, for instance:

> *Ahmet* uses the definition of the function, encourages students' visual thinking, establishes connections between the representations, and encourages students' flexibility in thinking of the function concept. *Burak* emphasizes procedures not the concept, engages students with the visual properties of the piecewise functions, does not link analogies to targeted concepts taught.

This process was repeated several times on different copies of the texts, and each occurrence brought in new codes and some of the previous codes were revised. As this process went on, the relationships between the units of meaning became clearer and, thus, a system of pattern-coding was employed to collect the units of meaning and themes under more general groups. Repetition of this second process led to the creation of more general categories which are presented as features of action-oriented and process-oriented teaching approaches in the coming section. As it was the manner in the analysis of observation data, interviews with the students were fully transcribed and then analyzed from the perspectives of action-process conceptions. In this process the researcher's field notes were also used to complement the interview transcripts. Additionally, descriptive statistics was conducted to analyze the students' test results.

Finally, the method of cross-case analysis (Miles & Huberman, 1994) was employed to explain the relationships between the teachers' instructional practices and the students' understanding of the piecewise function. A comparison between the sets of data was made in two ways. First, after an analysis of the instances where the teachers differed from each other, correspondingly different occurrences within the students' data were considered. Second, I identified the instances in which the classes of students displayed noticeable differences in understanding the piecewise functions, and then looked for the corresponding teaching inputs in the teachers' instructions.

Results

The results are presented in two parts. First, I illustrate the teachers' instruction of the piecewise function, and second, I examine the students' understanding of this concept.

Teaching Orientations

The two teachers differed considerably in their teaching of the piecewise function. Overall Ahmet's instruction could be described as a process-oriented teaching the key feature of which was that Ahmet prioritized the concept of piecewise function and its properties. He utilized concept definition as a cognitive tool to ensure that his students understand that a piecewise function produces an output for every input. He provided concept-driven, clear, and explicit verbal explanations to emphasize a single function process represented by more than one rule on the sub-domains. Connection between algebraic and graphical representations was another distinguishing aspect of Ahmet's process-oriented teaching. Ahmet's teaching also included features that would improve his students' procedural fluency in manipulating algebraic functions, such as calculating the images when the pre-images are associated with two piecewise functions. Ahmet's lesson on the piecewise function can be considered in two parts: the introduction and the development. During the introduction Ahmet illustrated the notion of piecewise function through a particular example through which he emphasized two ideas:

- A piecewise function is described by two or more rules that operate in particular sub-domains; yet the sub-domains make up the domain of the piecewise function, and the rules represent a single function.
- A piecewise function transforms every element of the domain to only one element of the co-domain as if it was a function described with a single rule.

The following citation illustrates how the teacher emphasized these ideas (**Episode A1).**

Ahmet: […] If a function is described by more than one rule we call it a 'piecewise function.' How can a function be described by more than one rule? It is strange, isn't it. Have a look at this example,

$$f: R \to R,\ f(x) = \begin{cases} 3x-1, x < -2 \\ x^2+1, -2 \le x \le 5 \\ 4x, x > 5 \end{cases}$$

> As you see the function $f(x)$ is defined from R to R; but it is described by three rules, $3x$-1, x^2+1, and $4x$. If you look at it closely you realise that each of these rules is given on restricted domain(s). That is, the function is described by "$3x$-1" when the value of x is less than -2 [and] by the x^2+1 when $-2\leq x\leq 5$. We call these sets the sub-domains of the function; existence of three sub-sets should not confuse you - these are the sub-domains which eventually make up the domain set, R; and the rules on these sub-domains represent the same function, the function $f(x)$. This is the nature of a piecewise function that differentiates it from those described by a single rule. Otherwise there is no difference between them. A piecewise function matches every element in the domain to only one element. [...] For example, if the input is less than -2 this function does the matching by means of this rule, $3x$-1[...]

In this episode the teacher, Ahmet, illustrates surface properties of the piecewise function – a piecewise function is described by more than one rule and each rule operates on a particular sub-domain - but also he goes beyond these surface properties and uncovers the meaning behind them. It is conjectured that the instances – "these are the sub-domains which eventually make up the domain and the rules on the sub-domains represent the same function" and "A piecewise function matches every element in the domain to only one element" are likely to facilitate students' construal of the situation as a single process transforming every input to an output.

In the development part of the lesson Ahmet continued to strengthen the students' understanding of the concept. In addition to abovementioned variables his teaching included three particular instructional inputs and these were:

- Uses a number line as an instructional aid (places the domain of a piecewise function on a number line). This strategy was employed when examining algebraic piecewise functions. The aim was to enhance students' visual ability so that they could recognize the conditions where an algebraic piecewise function does or does not represent a function – (later this strategy was used by one of Ahmet's students during the interview; see Demet's responses to algebraic situations).
- Establishes connections between algebraic and graphical representations of the piecewise functions. This was occurred in Ahmets' lesson as shifting back and forward between algebraic and graphical representations of the same function. For instance, on one occasion Ahmet gave his students an algebraic piecewise function, $f(x)=\begin{cases} x+1, & x\geq 3 \\ -2x, & x<0 \end{cases}$ and then he sketched, through discussing the issue with his students, its graph.
- Encourages students' flexibility to think of a situation from different perspectives.

The following episode illustrates how Ahmet enforced his students' flexibility to thinking of a graphical situation from different angles. The teacher examined, with the active participation of the students, why the graph (see Figure 1) did not represent a function on the set of **R;** and then he situated the problem into the context of piecewise function and gave the following explanation (**Episode A2**).

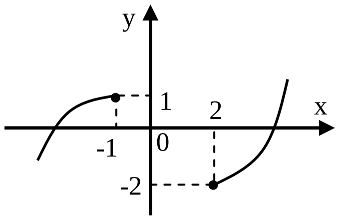


Figure 1. A representation of the graph of piecewise function derived from Ahmet's handout.

Ahmet: […] Let's have a look at the issue [problem] from another perspective. Let me see who are able to bring an alternative approach and who are not If I say this is actually a graph of function, do I confuse you? [*After a short silence some students got the point and suggested that the elements between -1 and 2 should be removed from the domain*] Yes, it does, but not on the set of R. We have to redefine the domain. How could we do that? It is quite obvious, look at the graph; it tells us what we should do. It covers this part and that parts of the x-axis [*Moves his finger on the sub-domains*]. Here is a graph made of two segments with two sub-domains; so what does it mean? It means that this is, in fact, a graph of piecewise function If we determine the domain set as $(-\infty, -1]\cup[2, \infty)$, this graph matches every element in this set to only one element [*illustrates the matching over the graph*].

The premise of this episode is the construction of a process of a piecewise function which was not initially there. It is suggested that Ahmet follows a pedagogically sound strategy; since he illustrates, first, the surface properties of the piecewise function – the segments of the graph and the sub-domains – in relation to each other; and then he forms a single domain by unifying the sub-domains and explicates how this function transforms elements from domain to co-domain over the graph. This sequence of teaching acts does not focus students' attention upon the visual properties of the graph – looking at a graph – but enforces them to understand the meaning behind the graph – looking through a graph.

In contrast to Ahmet, Burak employed an action-oriented teaching in that he emphasized rules, procedures, and the factual knowledge associated with the algebraic piecewise functions. Burak's teaching hardly involved explanation, discussion, or questioning that could help students understand the concept of piecewise function and its properties. The definition of function was not even an implicit referent in his verbal explanation. It appeared that Burak's instructional goal was to ensure that his students acquired competence in performing a procedure that involved selecting the right formulas to apply on each sub-domain. He utilized two strategies to achieve this: comparing input(s) with the extreme points of the sub-domains and providing examples and analogies from the everyday life. The Cartesian graph was absent in his teaching. Burak's teaching can be considered in two phases: the introduction and the development. Burak introduced the idea of piecewise function through an analogy "the weather condition" through which he associated the clothes worn under particular weather conditions, and he often referred to this analogy as the instruction developed (**Episode B1**):

Burak: […] Let me explain the logic of piecewise function with an example. I believe it will help you. […] We dress up according to whether condition, don't we? If it is cold, we put on thick clothes; if it is sunny and hot we dress up in thin and relaxing clothes. If the weather is rainy we put on raincoat. What are we doing here? We are dressing up according to a certain condition; the weather condition. The logic of piecewise function is the same; it is given with certain conditions. […] While making manipulation we select certain rules depending upon the numbers we give for *x*.

The teacher believes that the analogy would help the students understand the logic of the piecewise function; yet through the analogy he prepares students for a procedure that focuses upon the selection of the right formulas to apply on the sub-domains. The way he presents the analogy is far from creating in the students an image of a piecewise function which is defined by more than one rule but represents a single process transforming every input to an output.

The procedure suggested above became absolute focus of instruction during the development phase of the lesson. Burak illustrated how to make manipulations with the algebraic piecewise functions – calculating images when the pre-images are given and composing two piecewise functions at certain point(s). The students were active participants of the lesson in that they were always engaged with the algorithmic manipulations and as the instruction developed the students became quite competent in them. The lesson turned out to be a kind of enterprise in which the teacher assigned routine problems and the volunteers resolved them on the board. Burak's first task was typical:

Task 1: Given the function *f*: R→R,

$$f(x) = \begin{cases} \dfrac{3x+4}{9}, x < -2 \\ x^2 - 5, -2 \le x < 3 \\ 2^{x-4}, 3 \le x \end{cases}$$

what is the value of *f*(-5)+*f*(0)+*f*(4)?

The following episode illustrates key aspects of Burak's instruction in other similar situations (**Episode B2**):

Burak: [...] We call this type of function a piecewise function. What does this mean? It means that the rule of function changes in accordance with the numbers given for *x*. In other words, the function is described by different rules for different values of *x*. Look at it; when the value of *x* is less than -2 we are going to use this rule ($\frac{3x+4}{9}$).We are going to use x^2-5 when -2≤*x*<3. Remember the example I gave at the beginning [of the lesson]; we should dress up according to weather condition…the logic is the same …we are going

> to choose the formula according to the numbers we give for *x*. Let's find the value of *f*(0). 0 is bigger than -2 and less than 3; therefore we shall use the formula in the middle…take the square of 0 and then subtract 5. [*he then completes the manipulations*]

This episode is all about the procedure. Throughout the first few sentences Burak emphasizes a factual knowledge – a function could be described by different rules – nevertheless, in the following parts he brings no clarification to the basic ideas: the sub-domains (which make up the domain of the function) and a single function process behind the expressions. Apart from the words *'function'* and *'piecewise function'* he does not use fundamental terms, such as inputs, outputs, transformation or matching, which are critical to draw students' attention on the transformation that the function does from domain to co-domain.

The key features of each teacher's instructional approach are summarized in Table 1.

Table 1

Key features of the teachers' instruction of the piecewise function

Ahmet (Process-oriented Teaching)	Burak (Action-oriented Teaching)
• Teaches, mostly, the concept of piecewise function • Establishes connections between the representations • Uses the definition of the function as a cognitive tool to drive a solution to the problems • Encourages students to think of a situation from different perspectives • Uses a number line to encourage students' visualization of the domain set when the function is given algebraically • Provides concept-driven, clear and explicit verbal explanation to emphasize the concept	• Teaches, mostly, rules, procedures, and the factual knowledge associated with the algebraic piecewise functions • Does not establishes connections between the representations • Does not use the graphical representations in his teaching • Often provides analogies, but does not link it to the concept of piecewise function or uses it to illustrate the procedure – selection of the right formulas to operate on the sub-domains

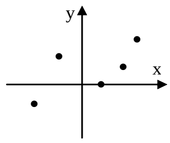
In summary, as illustrated in Table 1, the fundamental distinction between process-oriented and action-oriented teaching is grounded in the teachers' approaches to the essence of the concept. Ahmet not only illustrates the surface properties of the piecewise function but also discerns the meaning behind them. Unlike Ahmet, Burak mostly emphasizes rules, procedures, and the factual knowledge associated with the algebraic piecewise functions. I now consider the impacts of these two teaching orientations on student learning.

Learning Outcomes

The pre-test and the follow-up interviews indicated that the groups of students were largely comparable in their initial knowledge of function which was assessed on two scales: informal knowledge of functions and prerequisite skills. The former entailed an understanding of matching between the elements of two sets, an understanding of dependence between two varying quantities, an understanding of a relation (an implicit process) in a set of ordered pairs, and an interpretation of a transformation process represented by a (function) box with the algorithms inside. The latter assessed their ability to read a graph representing a real world situation, to read a table, and to make manipulations with the algebraic expressions. The test indicated, for example, that 67.9% of Class A and 77.8% of Class B gave correct answer to a question: “When you are asked to double the circumference of a circle, what would you do? Explain your reasoning,” by suggesting that the radius of the circle should be doubled because it is the only variable that can be manipulated and the perimeter of a circle depends on its radius.

After the instructional treatment the groups of students displayed noticeable difference in their conceptual understanding. The examination of the students’ learning draws upon their responses within the post-test and from the interviews with six selected students. Three items are considered:

Item 1: Consider the graph made of five discrete points. Does it represent a function? Give your answer with the underlying reasons.



Item 2: Given the functions *f: R→R* $f(x)=\begin{cases} x-5, & x\geq 5 \\ 4x+1, & x<3 \end{cases}$ and

g: R→R $g(x)=\begin{cases} x-3, & x\geq 4, \\ -x+4, & x<4 \end{cases}$, what is the value of ($gof$)(1)=?

Item 3: Identify whether or not the relation $y=\begin{cases} 2x-3, & x\geq 5 \\ -x+4, & x<5 \end{cases}$ represents a function on R ($y=f(x)$). Give your answers with the underlying reasons.

Table 2

Students’ understanding of the piecewise function in a graph made of five discrete points.

Answers to Item 1 / Classes	Class A		Class B	
	Frequency	Percent	Frequency	Percent
Function (refers to definition, specifies the domain)	17	60.7	8	29.6
Not a function (concerning the univalence)	2	7.1	2	7.4
Not a function (continuity restriction)	4	14.3	12	44.4
Not a function (other reasons)	2	7.1	1	3.7
No response	3	10.7	4	14.8
Total (n / percentage)	28	100	27	100

Note: Because of the rounding error percentages seem to add up to 99.9%.

Table 2 summarizes students’ responses about the piecewise function associated with the graphical representation, namely Item 1. Those who considered that the graph was a function specified the domain on the *x*-axis and illustrated the transformation over the graph; and in this respect Class A outperforms the Class B in a ratio of 2 to 1. Students who reported that the graph was not a function were categorized at three levels. The largest group within each class displayed a continuity misconception. These students joined the points with curves or broken-lines and then claimed that the graph they had sketched represented a function. Notice that the number of students in Class B who revealed ‘continuity misconception’ is three times as much as the number of students in Class A who did so.

The interviews complemented the class differences identified through the post test. It is conjectured that two students (Okan and Demet) from Class A and one (Aylin) from Class B had a process conception. These students recognized the process of piecewise function defined on the sub-domains and illustrated how this process does a ‘one-to-one’ matching from domain to co-domain. Okan’s answer was typical.

> *Okan:* We must specify the domain [*marks the inflection of the graph on the x-axis*]. Look every point [*refers to the segment of the graph*] matches an element in the domain to only one element in the co-domain [*illustrates matching over the graph*]. I mean if the domain contains only these five elements [*labels the points as a, b, c, d, e*], this graph represents a function.

Two students, one from each class (Erol from Class A and Serap from Class B), were apparently in transition towards a process conception of function, because they indicated partial understanding of the situation. They rejected the graph arguing that the graph did not satisfy the univalence condition; however they were unable to recognize the process of piecewise function defined on the split domain. In their view the only way to identify a situation where a graph could represent a function was joining the points. The third student, Belgin, was considered at the action level because she rejected the graph but could not give a reasonable explanation as to why it was not a function. On my probing she acted with the concept image (Vinner, 1983) and disclosed a misconception:

> *Belgin:* In my view, the graph of function must be continuous line or a curve. I should join them. [silence for a few seconds] I have not seen any graph like this, umm, I have to join these points in some way.

The groups differed again in their understanding of the concept in the algebraic context. The majority in both classes (Class A: 89.3% and Class B: 74.1%) composed two piecewise functions at x=1 (Item 2) – a learning outcome considered as the indicator of an action conception of piecewise function. The groups were equally competent (Class A: 78.6% and Class B: 77.8%; see Table 3) in identifying the situation $y = \begin{cases} 2x-3, x \geq 5 \\ -x+4, x < 5 \end{cases}$ as a function; however the students of Class A displayed overwhelming superiority over those in Class B in examining the situation with regard to concept definition (Class A: 67.9% and Class B: 25.9%) – an indicator of a process conception of piecewise function.

As illustrated in Table 3, half of Class B students simply marked the situation as ‘function’ or indicated satisfaction with the given rule (algebraic expressions). Those who indicated satisfaction with the rule inserted a couple of elements into the function and made manipulations, or they provided statements indicating that it was a function because it was given with an algebraic formula.

Table 3
Students' understanding of the notion of piecewise function in algebraic representation.

Answers to Item 2 / Classes	Class A		Class B	
	Frequency	Percent	Frequency	Percent
Function (refers to definition)	19	67.9	7	25.9
Not a function (a statement or refers to rule)	3	10.7	14	51.9
Not a function	2	7.1	3	11.1
No response	4	14.3	3	11.1
Total (n / percentage)	28	100	27	100

In the interviews, the six students worked out the image of x=1 under the composite function (Item 2) without any confusion caused by the selection of the appropriate formulas for the sub-domains. However, in responding to Item 3 two students (Okan and Demet) from Class A and only one student (Aylin) from Class B displayed a process conception; these students construed the relation(s) as a single process and examined them with regard to concept definition. Demet's answer was typical. She considered that the expression given in Item 3 was a function and indicated, "it is a piecewise function. Whatever we put into the x, we get out an image for that input". When "<" was replaced by "≤" in the sub-domain below, Demet recognized that the expression was no longer a function, emphasizing that it now produced two images for a single input, 5. Her response to a follow-up question, $f(x)=\begin{cases}2x, x\geq 7\\ -3x, x<4\end{cases}$ defined on **R**, included:

> *Demet:* Yes, it is a function. [*silence for a few seconds*]. Let me explain it on the number line [*places the sub-domains on a number line as in Figure 2*]. Umm, just a second, oh sorry, no, no; it is not a function, the numbers between 4 and 7 have not been matched to-

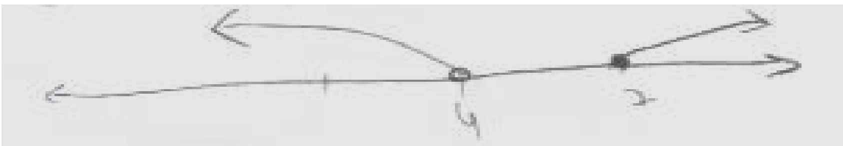


Figure 2. Demet's written explanation to an algebraic relation defined by two rules on the sub-domains.

This is a particular instance which shows the positive impacts of process-oriented teaching on students' understanding of the piecewise function. The strategy that the student uses had been recommended by the teacher of Class A (Ahmet), and apparently it is facilitating the student's interpretation of the process of piecewise function in the light of concept definition.

It is conjectured that three students, Erol (Class A), Serap (Class B) and Belgin (Class B) possessed an action conception of piecewise function; these students obtained the image of x=1 under the composite function, fog (Item 2), through step-by-step calculations; yet neither of them was able to examine the relation(s) in light of concept definition. Erol and Serap were reliant upon algebraic manipulations to evaluate the situation(s) whilst Belgin acted with the concept image. For instance, Serap's response to initial task (Item 3) was: "If the number is equal to or bigger than 5, I use the above formula and calculate the image of that number.” As I replaced “<” by “≤” in the sub-domain below she came through with a similar answer without recognizing that the expression produces two images for an input, 5. This was not a trivial oversight; although she was urged to examine the relation, $y=\begin{cases} x^2+3, x>3, \\ 2x+4, x<1 \end{cases}$ defined on **R**, in relation to the concept definition she said:

> *Serap:* It is a piecewise function. How can I explain it? I mean, when we give a number for *x* bigger than 3, we work out its image by using the above formula. If the input is less than 1, we use the formula below.

Table 4 summarizes the interviewees’ development of the piecewise function in the algebraic and graphical situations.

Table 4
The interviewees’ development of the piecewise function through graphical and algebraic representations

Representations	Class A			Class B		
	Okan	Demet	Erol	Aylin	Serap	Belgin
Graphical representation	P	P	A→P	P	A→P	A
Algebraic representation	P	P	A	P	A	A

This table suggests that two students from Class A and one from Class B possess a strong process conception of piecewise function in the algebraic and graphical contexts. It is conjectured that these students might have made some progress towards an object conception in the algebraic situation because they are able to obtain a new function by combining two or more processes on the sub-domains and think of the new function as a single unit. Two students, one from each class, illustrate that their conceptions of piecewise function in the graphical situation is in transformation from an action to a process conception whilst their conception in an algebraic context is action oriented; with a student possessing an action conception in both context.

Discussion and Conclusion

Simply defined, teaching refers to instructional acts taken to help students construct knowledge. It is a complex cognitive skill delivered in an ill-structured and dynamic environment (Leinhardt, 1988). The process of teaching draws upon more than one type of knowledge including, for instance, subject-matter knowledge (Even, 1990; Ball, 1991), pedagogical content knowledge (Shulman, 1986), and knowledge of lesson structure (Leinhardt et al., 1991); and it requests a variety of social skills for classroom management. Learning is a cumulative process that an individual develops through interacting with the internal or external stimuli. It is a mental process which occurs entirely in the mind of individuals, and this mental processing is still unknown, for the most part, to the educators (Eisenberg, 1991). The mediating process between teaching and learning is open to influence of many internal and external factors that may include the individuals’ cognitive capability, their attitudes towards mathematics, parental involvement in students' education, and the type of society in which the students live. The difficulty in controlling the influence of these factors does not permit an explanation of the relationships between classroom teaching and the student learning in the sense of cause-effects relations. In this article, it is also not claimed that there exist a one-to-one-relationship between teaching practices and students' learning.

Nevertheless, the provided evidences suggest that teaching practices that differ in their approaches to the essence of mathematical concepts are likely to produce qualitatively different learning outcomes in the students. It appears that Ahmet's process-oriented teaching have encouraged his students to develop a process conception of piecewise function whilst Burak's action-oriented teaching largely constrained his students' understanding of the concept to an action conception of piecewise function. Obviously, the development in each group of students' understanding cannot be explained by any one particular aspect of the teachers' instruction; instead it can be best construed as the full impact of teaching inputs that make up action-oriented and process-oriented teaching approaches (see Table 1). In addition, what the students had learned in other lessons on functions might have positively or negatively affected their acquisition of the piecewise function. Furthermore, some students might have got support from their schoolmates or parents and this might have affected their learning of piecewise function. For instance, Aylin, from Class B, indicated a process conception of piecewise function although she received, mostly, action oriented teaching practices during the classes. This case eliminates establishing one-to-one relations between the teachers' instructional practices and their students' learning.

Having said this, to illustrate the distinction between the two teaching orientations and its consequent learning outcomes, I consider again the key features of these teaching orientations. As has been seen in Episode A1 the cognitive focus of Ahmet's instruction is on the idea of piecewise function. He not only illustrates the visual properties of the piecewise function but also continually engages his students with the process of piecewise function associated with the rules on the sub-domains. Ahmet uses the definition of the function as a cognitive tool and addresses that a piecewise function transforms every element of the domain to a unique element in the co-domain as if it was a function defined with a single rule over the whole domain. To encourage his students' visualization of this transformation he provides an instructional aid – places the domain set on a number line. The effectiveness of this process-oriented teaching can be seen in the students' data. In the post test, two thirds of Class A students indicated a full understanding of the concept in the algebraic form – a feature confirmed through the interviews with the two Class A students. 61% of Class A students established a process of piecewise function in a graph made of five discrete points, and again the possession of this process conception of piecewise function was revealed during the interview when two of his students did so.

Unlike Ahmet, Burak mostly engages his students' with the rules, procedures and the factual knowledge (see Episode B1). His instructional goal appears to ensure that his students acquired procedural knowledge which includes selection of the right formula to apply on each sub-domain. To ensure this Burak gives his students a strategy which entails comparing inputs with the extreme points of the sub-domains. He often provides analogies – clothes worn under certain whether conditions (see Episode B1) – yet the instructional goal is again to emphasize the procedures. Burak does not use graphical representations in his teaching of the piecewise functions. Additionally, all the graphs he used in his teaching on function unit were smooth and continuous graphs, and this was itself a limitation which could, and did so, encourage a continuity misconception amongst his students. We can see the negative impacts of this action-oriented teaching on his students' learning. In the post test, 74% of Class B students composed two algebraic piecewise functions at a point – indicator of an action conception of piecewise function – but only 26% of them indicated a full understanding of the concept in an algebraic form. In the interview, only one of Burak's students interpreted a process of algebraic piecewise function and examined it in light of concept definition. 44% of Class B students revealed a continuity misconception joining five discrete points by broken lines or curves.

The given evidence suggests that teachers have a considerable role to play in students' knowledge construction. They perform this role by creating opportunities in which students construct their own knowledge. The evidence provided in this article suggests that to help students acquire epistemologically correct and conceptually rich knowledge of piecewise function teachers should prioritize the concept itself. The focus of teachers' instruction should be the idea of piecewise function while the related rules and procedures are illustrated as part of the routine. This can be achieved, as we see to some extent in Ahmet's teaching, by drawing students' attention upon the notion of piecewise function, keeping them engaged with the concept throughout the lesson and encouraging them to reflect collectively upon what has been taught and learned. In this respect, the presented results incorporate the study of Cobb et al. (1997) which indicated that the teachers' efforts to make the concept an object of discussion among the students and to enforce their collective reflections upon the concept prompted students' understanding of arithmetical concepts from an action-process conception to an object conception.

The evidence presented here, for Ahmet's teaching, indicate that to help students develop a meaningful understanding of the piecewise function teachers should allow their students to experience the concept across the representations available. This observation validates once more a pedagogical principle that a meaningful understanding of a mathematical concept can be attained when a variety of representations have been developed and the functional relationships are established amongst them (Goldin, 2001). On the other hand, Burak's teaching and his students' performances compromise Schwarz and Dreyfus' (1995) observation that using representational systems in isolation and relying upon a single representation could produce ill-structured knowledge in which properties of the function concept are attributed to the formal representations (algebraic expressions or graphs) and not to the concept itself.

Explanation is at the heart of education (Leinhardt, 1988). Particularly, in the traditional teaching-learning environment spoken language is one of the basic tools that the teachers utilize to illustrate the mathematical concepts to their students. The evidences provided in this paper suggest that to promote students' conception of piecewise function from an action to a process conception teachers should use the definition of the function concept as a cognitive tool. They should provide concept-driven, clear and explicit language. The language is concept-driven in that the teachers continually refer back to the definition of the function concept in their verbal explanations while resolving problems about piecewise functions. It is clear and explicit in that the teachers' verbal explanations emphasize the process of a piecewise function behind the representations (algebraic expression or Cartesian graph) that transforms elements from domain to co-domain (Bayazit, 2006). Teachers are suggested to use continually in their verbal explanations some crucial terms – such as domain, co-domain, input, output, transformation – so that they could enhance their students' conception of a piecewise function as a single process transforming every input to an output.

It is generally believed that the instructional analogies could facilitate students' learning of mathematical concepts (English, 1997). This would have a psychological ground because instructional analogies could act like a benchmark to which students could link what they learn. Appropriate analogies could assist students to reduce the complexity of mathematical concepts and hold them in a relatively small space in mind (English, 1997). Nevertheless, the present study indicated, as it is seen in Burak's case, that provision of analogies cannot facilitate a meaningful learning unless they are (re)organized and presented in a way that illustrates the essence and the properties of the targeted concept to the learners. The teachers are suggested, therefore, to give attention to two crucial features of analogies. First, the source analogue that they use should satisfy the content validity in that the analogue must have epistemological power to represent the essence and the properties of the targeted concept – the notion of function. Second, they should illustrate (or encourage their students to find out) the structural relations between the analogues and the targeted concepts. They might use analogies, as Burak did, to explain surface properties of the piecewise function and the associated procedures. Yet, this might intensify the importance of procedures for the students and, consequently, confine their understanding of the concept to mechanical manipulations.

References

Askew, M., Brown, M., Rhodes, V., William, D., & Johnson, D. (1996). *Effective teachers of numeracy*. London: King's College.

Ball, D. L. (1991). Research on teaching mathematics: Making subject-matter knowledge part of the equation. In J. Brophy (Ed.), *Advances in Research on Teaching* (pp. 1-48). Greenwich, CT: JAI Press, 2.

Bayazit, İ. (2006). The relationship between teaching and learning through the context of functions. *Unpublished PhD dissertation*, University of Warwick, UK.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*(3), 247-285.

Brophy, J., & Good, T. L. (1986). Teacher Behavior and student achievement. In M. C. Wittrock (Ed.), *Handbook of Research on Teaching* (pp.328-375), New York: Macmillan.

Cobb, P., McClain, K., & Whitenack, J. (1997). Reflective discourses and collective reflection. *Journal for Research in Mathematics Education, 28*(3), 258-277.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the Limit Concept: Beginning with a Coordinated Process Scheme. *Journal for Mathematical Behavior, 15*(2), 167-192.

Dubinsky, E. (1991). Reflective abstraction. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (pp.95-123). Dordrecht: Kluwer Academic Publishers.

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy* (85-107). United States of America: Mathematical Association of America.

Eisenberg, T. (1991). Function and associated learning difficulties. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (pp.140-152). Dordrecht: Kluwer Academic Publishers.

English, L. D. (1997). Analogies, metaphors, and images: Vehicles for mathematical reasoning. In L. D. English (Ed.), *Mathematical Reasoning: Analogies, Metaphors, and Images* (p. 3-18). New Jersey: Lawrence Erlbaum Associates Inc.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics*, *21*(6), 521-544.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27*(4), 403-434.

Goldin, G. (2001). Systems of representations and the development of mathematical concepts. In Cuoco, A. A., & Curcio, F. R. (Eds.), *The role of representation in school mathematics* (p. 1-23). Reston. NCTM Yearbook.

Good, T., & Grouws, D. (1977). Teaching effects: A process-product study in fourth grade mathematics classrooms. *Journal of Teacher Education*, 28, 49-54.

Helmke, A., Schneider, W., & Weiner, F. E. (1986). Quality of instruction and classroom learning outcomes: The German contribution to the IEA classroom environment study. *Teaching and Teacher Education, 2*(1), 1-18.

Leinhardt, G. (1988). Expertise in instructional lessons: An example from fractions. In D. A. Grouws, & T. J. Cooney (Eds.), *Perspectives on Research on Effective Mathematics Teaching* (pp.44-66). Hillsdale, NJ: Lawrence Erlbaum.

Leinhardt, G., Putnam, R. T., Stein, M. K., & Baxter, J. (1991). Where subject knowledge is matter. In J. Brophy (Ed.), *Advances in Research on Teaching: Teacher's Knowledge of Subject Matter as it Relates to Their Teaching Practice* (v2, pp.87-113). London: JAI Press Inc.

Markovits, R., Eylon, B. S., & Brukheimer, M. (1986). Functions today and yesterday. *For the Learning of Mathematics*, *6*(2), 18-28.

Merriam, S. B. (1988). *Case study research in education: Qualitative approach.* London: Jossey-Bass Publishers.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis (An expanded sourcebook).* London: Sage Publications.

Phillips, N., & Hardy, C. (2002). *Discourse analysis: investigating processes of social construction*. United Kingdom: Sage Publications Inc.

Pirie, S., & Kieren, T. (1992). Creating constructivist environments and constructing creative mathematics. *Educational Studies in Mathematics, 23*, 505-528.

Schwarz, B., & Dreyfus, T. (1995). New actions upon old objects: A new ontological perspective on functions. *Educational studies in mathematics*, 29(3), 259-291.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification-the case of function. In G. Harel, & E. Dubinsky (Eds.), *The Concept of Function Aspects of Epistemology and Pedagogy* (pp.59-85). USA: Mathematical Association of America.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*, 4-14.

Tall, D. O. (1999). Reflections on APOS theory in elementary and advanced mathematical thinking. In O. Zaslavsky (Ed.,), *Proceeding of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (v1, pp.11-118). Israil: Haifa

Tobin, K., & Capie, W. (1982). Relationships between classroom process variables and middle-school science achievement. *Journal of Educational Psychology*, 74, 441-454.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology, 14*(3), 293-305.

Weinert, F. E., Schrader, F-W., & Helmke, A. (1989). Quality of instruction and achievement outcomes. *International Journal of Educational Research, 13*(8), 895-914.

Author

İbrahim Bayazit, PhD, Erciyes University, Kayseri, TURKEY; e-mail: ibayazit@erciyes.edu.tr