LOOK ACADEMIC PUBLISHERS
OPEN ACCESS

IEJME — MATHEMATICS EDUCATION
2016, VOL. 11, NO. 10, 3511-3519

Students'Awareness on Example and Non-Example Learning in Geometry Class

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ABSTRACT

The main goal of the study reported in our paper is to characterize teachers' choice of examples in and for the mathematics classroom. Our data is based on 54 lesson observations of five different teachers. Altogether 15 groups of students were observed, three seventh grade, six eighth grade, and six ninth grade classes. The classes varied according to their level—seven classes of top level students and six classes of mixed—average and low level students. In addition, pre and post lesson interviews with the teachers were conducted,and their lesson plans were examined. Data analysis was done in an iterative way, and the categories we explored emerged accordingly. We distinguish between pre-planned and spontaneous examples, and examine their manifestations, as well as the different kinds of underlying considerations teachers employ in making their choices, and the kinds of knowledge they need to draw on. We conclude with a dynamic framework accounting for teachers' choices and generation of examples in geometry class

KEYWORDS
Example, non-example, geometry class

ARTICLE HISTORY
Received 22 July 2016
Revised 30 October 2016
Accepted 12 November 2016

Introduction

Examples are an integral part of mathematical thinking, learning and teaching, particularly with respect to conceptualization, generalization, abstraction, argumentation, and analogical thinking. By examples we mean a particular case of a larger class, from which one can reason and generalize. In our treatment of examples, we also include non-examples, that are associated with conceptualization and definitions, and serve to highlight critical features of a concept; as well as counter-examples that are associated with claims and their refutations. Both non-examples and counter-examples can serve to sharpen distinctions and deepen understanding of mathematical entities. It should be noted that examples may differ in their nature and purpose. An example of a concept (e.g., a rational number) is quite different in nature from an example of how to carry out a procedure (e.g., finding the least common denominator).

Moreover, the purpose for presenting an example may vary. Thus, a teacher may illustrate how to find a common denominator of two proper fractions for adding fractions, or s/he may illustrate it as a basis for generalizing the procedure to algebraic fractions in order to be able to solve more advanced equations

Studies on how people learn from worked-out examples point to the contribution of multiple examples, with varying formats (Atkinson, Derry, Renkl and Wortham 2000). Such examples support the appreciation of deep structures instead of excessive attention to surface features. Studies dealing with concept formation highlight the role of carefully selected and sequenced examples and non-examples in supporting the distinction between critical and non-critical features and the construction of rich concept images and example spaces (e.g., Vinner 1983; Zaslavsky and Peled 1996; Petty and Jansson 1987; Watson and Mason 2005). In spite of the critical roles examples play in learning and teaching mathematics, there are only a small number of studies focusing on teachers' choice and treatment of examples. Rowland, Thwaites and Huckstep (2003) identify three types of novice elementary teachers' poor choice of examples: choices of instances that obscure the role of variables (for example, in a coordinate system using points with the same values for both coordinates); choices of numbers to illustrate a certain arithmetic procedure when another procedure would be more sensible to perform for the selected numbers (for example, using 49×4 to illustrate conventional multiplication); and randomly generated examples when careful choices should be made. These findings concur with the concerns raised by Ball, Bass, Sleep, and Thames (2005) regarding the knowledge base teachers need in order to carefully select appropriate examples that are useful "for highlighting salient mathematical issues" (p. 3). bviously, the choice of examples in secondary mathematics is far more complex and involves a wide range of considerations (Zaslavsky and Lavie 2005; Zodik and Zaslavsky 2007; Zaslavsky and Zodik 2007). This paper provides a close look at the underlying considerations that experienced secondary school teachers employ.

The specific choice of examples may facilitate or impede students' learning, thus it presents the teacher with a challenge, entailing many considerations that should be weighed. Yet, numerous mathematics teacher education programs do not explicitly address this issue and do not systematically prepare prospective teachers to deal with the choice and use of instructional examples1 in an educated way. Thus, we suggest that the skills required for effective treatment of examples are crafted mostly through one's own teaching experience (Leinhardt 1990; Kennedy 2002). It follows, that there is much to learn in this area from experienced teachers. Our study proposes to make a step towards learning from experienced teachers—their strengths and difficulties associated with exemplification in the mathematics classroom.

The Method of Research

In our research, we had two preliminary stages in analyzing teachers' choices of examples, corresponding to two dimensions: Conditions for an example (that is, whether a particular object/case qualified as an example according to our criteria), and mathematical correctness (that is, whether the example satisfied what it was intended/supposed to from a mathematical perspective). This enabled us to focus on the collection of mathematically correct examples for further analysis and characterization according to other categories that emerged as we repeatedly looked into the data, in the spirit of the grounded theory approach (Strauss and

Corbin 1998). We report on these categories in the findings section. One of the insights we gained as we examined the data relates to the unit of analysis. It turned out that it made a lot of sense to analyze the underlying considerations that led to a particular example, rather than to try to characterize an example in itself. These considerations reflected not only the mathematics, but also the pedagogy that was employed, including the teacher's goals, available tools, interpretation of the situation, etc.

In order to provide a stronger and unbiased picture of the data, and to better understand the phenomena that were examined, we used some simple statistics following Miles and Huberman (1987) and Wiersma (2000). There is no claim about generalizing beyond the scope of the study. However, given the limited number of studies that examined teachers' use of examples in and for their classrooms, this analysis may help form some hypotheses for future research.

For internal consistency we followed Wiersma (2000), who claims that "If two or more researchers independently analyze the same data and arrive at similar conclusions, this is strong evidence for internal consistency" (p. 211). Thus, for each stage that required some sort of coding according to a classification system that we applied, we had two researchers code independently at least 15% of the relevant data. In all cases we got at least 90% agreement, with no discussions between these researchers. In addition, in cases of that were vague, the validity was enhanced by stimulated recall interviews with some of the teachers, in which they were asked to reason about their choice of examples and react to the researchers' interpretations.

Result and Findings

Examle and Non Example Learning Process

Provide guided practice with whole class or small group or partners

- Put a blank Frayer Model transparency on the overhead/IWB and distribute blank Frayer Model sheets to partners.
- Write the key content-specific word in the middle of the graphic organiser and ask students to do the same.
- Tell students that they will complete the graphic organiser together as they read the text. Before reading the text, provide clear "student-friendly" definitions of the key content-specific word and any other key vocabulary and have students quickly preview the selection, examining illustrations, headings, subheadings and diagrams. Previewing should take no longer than 1–2 minutes. Ask students what they think they will learn in the selection. Allow no more than 3–5 minutes for this discussion.
- Have students read the first part of the text with their partners.
- After students have read the first section of the text, work as a class to complete any part of the Frayer Model graphic organiser that can be finished based on that section. Ask students to tell *why* the terms they identify are examples and non-examples of the selected concept/word.
- Read the next section of text and continue to add to the graphic organiser.

Provide independent practice

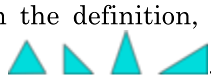
When students are proficient with the process, have them continue to work in partners, reading and adding to their graphic organisers. Monitor student work carefully and provide scaffolding and feedback as needed.

Generalisation

Discuss with students how might writing down examples and non-examples of words and using the graphic organiser help you learning the meanings of words in this and other KLAs.

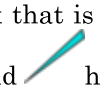
Example and Non Example of Geometry Concepts

Often enough, elementary school children will identify these figures  as "triangles" even though they can say that triangles have three sides. Why? Partly because children (and, for the most part, adults, too) learn words more from context than from definitions and explanations. Especially when we have no other word for this shape, we choose a best-fit category. These shapes look more like the concept image of triangles that children build up from examples like  than they look like any other shape for which the child has a name, so that is the category the child lumps them into.

By contrast, it is common enough for children not to recognize  as a triangle, despite its fit with the definition, because it is visually so different from the common examples . Similarly, people often enough refer to as an  "upside down triangle" (even though nothing about the definition of a triangle specifies what way it must sit) because the common image of a triangle sits on its base.

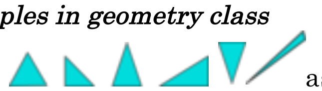
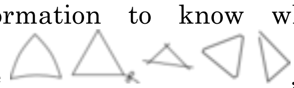
Examples help people "get" an idea in the first place, or extend or clarify an idea. But examples, as you have seen, can also create misunderstanding. This article shows ways in which examples are essential, risks of poorly chosen examples, ways to make the best of examples, and some limitations of examples.

Creating clear examples in geometry class

Clear examples illustrate the essential elements of an idea without distracting the learner with irrelevant patterns. The picture  limits the learner's image-of-triangle with three irrelevant patterns: all figures sit comfortably on a horizontal side at their bottom (triangles don't have to sit that way); all figures have symmetry or right angles or both (triangles are more varied); and all figures can be packaged in a box that is roughly as long as it is wide (triangles can be "extreme"). Including  and  helps the learner see the essential idea (three sides) and not inadvertantly include irrelevant ideas (orientation on the page, symmetry, or extreme skinniness).

Although those extra examples can correct wrong first impressions after they've occurred, presenting those examples first helps avoid students jumping to wrong conclusions. It takes more work to correct misunderstandings than to avoid them. Order matters. The first examples are especially influential when you are teaching without talking ("silent teaching"), and the examples you use are your only way of communicating. Those first three or four examples must contain enough information to help students not jump to wrong conclusions.

Non-examples in geometry class

Even with  as examples, a learner does not have enough information to know what is not a triangle. Selected non-examples, like , help focus attention on details that might otherwise be missed. The "three sides" must be straight, not curved; there can be no extra frills or bows or hanging-over bits of line (line segments must intersect only at their endpoints); the "points" can't be "cut off" (the shape is bounded by only three segments); the figure must be closed (all endpoints must be joined).

These non-examples were selected to be "near-misses," very close to the image people have of triangles. When children give verbal descriptions of triangles, they often mention "three lines" or "three corners," but omit the details that eliminate even fairly distant misses, like , which may sometimes be useful non-examples to help children improve their verbal descriptions.

Discussion

In everyday conversation, definitions are of little real help. Try to think, for example, how you would define "chair" to include all the different kinds of objects, wooden, plastic, stuffed, formal, etc., that are "chairs," and how to exclude superficially similar objects that are not chairs. Or think how to define "cat." Alternatively, imagine that you did not know these words; then look in a dictionary to see how much you must already know in order to understand the definition! Finally, think how little that definition really contains of the "cat" in your head. Definitions are not easy routes to meaning, even for adults, until one already has a fair idea what the word means from use in context—that is, from examples! It's not uncommon for adults to notice, when asked (perhaps by a child) the meaning of a word that they've long understood and used, that they don't really know, and have to look it up. For casual use, context and experience are enough to give us "the general idea" of a word, and make it useful even if we can not give a definition.

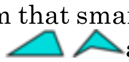
In a way, examples are bits of context -- ways to give information other than "saying what the word means" -- allowing children to acquire vocabulary in school a bit more the way they do out of school, at which they are so adept. Examples allow teachers to use a word communicatively until students are able to use it as well. Teachers can use the word rather than explaining it because the example provides the context and carries the meaning. Only then, when the students already have a rough meaning from communicative use in context can one effectively clarify the meaning formally with other words, through discussion and/or definition.

But examples are not enough. Even after a rough idea is acquired from examples, extreme examples, and non-examples, it is still the case that definition is needed. It is easiest to show why by example, using a whimsical category called "smanglings." Imagine developing this category inductively, through examples.

Examples:

Each of these is a smangling: 

Figure 1. Example in smangling concept

So far, it would seem that smanglings are squares of any size or orientation. When we learn that these  are also smanglings, we realize that the category is broader, and might decide that smanglings are probably four-sided shapes of any kind. But even these  are smanglings. So, now what do smanglings appear to be?

For pedagogical purposes – maybe for any purposes other than sheer perversity – the order in which the examples were presented was terrible. In order for examples not to be misleading (as we deliberately were in this case), it is important that they be as varied as the category permits (so, not just squares, not even just four-sided figures, if in fact, more variety is permissible). As stated above, it is often (but not always) the case that the variation should come quite early, so that the first impressions are not misleading, as they were here.

Extreme examples:

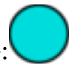
We cheated. This, too, is a smangling: 

Figure 2. Extreme example in smangling concept

It is tempting, now, to conclude that smanglings can be any closed two dimensional shape. But it is not logical to draw that conclusion, just as we could not (logically) conclude, before the circle was included, that smanglings could be any polygon. The correct answer to "What is a smangling?" is "We can't tell, because we haven't seen anything yet that isn't a smangling."

Non-examples:

So, here are a few non-smanglings: 

Figure 3. Non-example in smangling concept

Without these non-examples, how could we have guessed that color mattered? In fact, it would also have been illogical to make such a guess, as there was no evidence one way or the other.

Extreme non-examples:

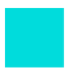
Here is another non-smangling:  Apparently the border must be visible

Figure 4. Exteme non-example in smangling concept

At this point we do have a lot of information. Let's see how far it gets us.

Testing the concept:

Which of these is a smangling?

Figure 5. Testing the concept in smangling

Conclusion

We can be fairly sure that a and d are smanglings; we can be equally sure that g is not. But what about the rest? Case c is the right color and has a black border, but we don't know whether decorations on the inside are allowed. No non-example rules them out, but no example shows them. This exercise (because it is purely inductive, without definitions) is more like science than mathematics: we can hypothesize pending further testing, but we cannot decide. It seems likely that case e should be ruled out as it has no black border, but case f is harder to rule out. It has the black line, and though it is not the right color on the "inside," it can't be because it doesn't have an inside! We have no evidence about this case. It certainly doesn't fit the examples, but it isn't ruled out by the non-examples, either. We have the same problem with case b. It is the right color on the inside, and the border is visible, but we've learned that color matters. Maybe the color of the border matters, too, and we simply don't know.

In fact, no matter how numerous and varied our examples and non-examples are, unless they are exhaustive (i.e., the set of smanglings is finite, and we have encountered every one of them as an example), examples alone are insufficient to allow us to decide all cases, because they provide no way of knowing whether or not some perverse exception lurks among the cases that have not been seen. But the examples -- and especially the task of trying to choose among the unknowns and then defend that choice -- make it much easier to understand a definition now than it might have been at the start. One advantage for students of encountering this meta-mathematical idea is that it helps motivate what otherwise often seems like bizarre over-particularity in the wording of definitions. There is a lot we must say to define a smangling in a way that allows us to decide, definitively and without question, which of the unknowns is and isn't a smangling.

Disclosure statement

In accordance with IEJME-Mathematics Education policies and procedures and my ethical obligation as a researcher, I am reporting that I have disclosed those interests fully to IEJME-Mathematics Education, and I have in place an approved plan for managing any potential conflicts arising from my article.

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Appendices

Appendix 1. Examples using the Frayer Model

<table>
<tr><td>Definition (in own words)
• a clause which expresses a complete thought</td><td>Characteristics
• has a subject and verb
• can stand alone as a sentence</td></tr>
<tr><td colspan="2">INDEPENDENT CLAUSE</td></tr>
<tr><td>Examples (from own life)
• She laughed
• I love to read</td><td>Non-Examples
• Left our house
• Because she was late
• In the room</td></tr>
</table>

<table>
<tr><td>Definition (in own words)
• A whole number with only two different divisors (factors), 1 and itself</td><td>Characteristics
• 2 is the only even prime number
• 0 and 1 are not prime
• Every whole number can be written as a product of primes</td></tr>
<tr><td colspan="2">PRIME NUMBER</td></tr>
<tr><td>Examples (from own life)
• 2,3,5,7,11,13,...</td><td>Non-Examples
• 4,6,8,9,10,12,14</td></tr>
</table>

Appendix 2. Frayer Model Graphic Organiser

<table>
<tr><td>Definition (in own words)</td><td>Characteristics</td></tr>
<tr><td colspan="2">TARGET WORD/ CONCEPT</td></tr>
<tr><td>Examples (from own life)</td><td>Non-Examples</td></tr>
</table>