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Gas-dynamic Waves and Discontinuities

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ABSTRACT

In this paper we examine the history of the studying the dynamic compatibility conditions for gas-dynamic discontinuities, which determine the ratio between values of the gas-dynamic variables before the discontinuity and right behind him. The concepts of a shock wave, shock and the shock polar are introduced. The formation of ideas about the shock waves as a narrow region with abrupt changes in gas-dynamic parameters is shown with a staged scientific studies as an example. The relationship between the physical nature of gas-dynamic discontinuities and the appearance of singularities in solutions of the Euler equations for an ideal gas is shown. Burgers equation, which allows to simulate the shock waves is discussed. The article can serve as a brief introduction to the theory of gasdynamic discontinuities. It proposes the modern idea of gasdynamic discontinuities as the features arising in the solution of hyperbolic partial differential equations.

KEYWORDS
Shock waves; gas-dynamic discontinuity; conditions of dynamic compatibility; shock polar; propagation speed of discontinuity

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Introduction

Shock waves model $(\bar{D})$ is a surface of first degree mathematical discontinuity, during the transition through which the gas-dynamic variables are discontinuous $[f] = \hat{f} - f \neq 0$. It is traditional to separate the rarefaction waves $\bar{R}_r$ (non-stationary simple isentropic Riemann wave) and $\bar{\varpi}_r$ (stationary centered Prandtl-Meyer wave) in which the ratio of static pressures $J_r \equiv \hat{p}/p < 1$ and wave compaction (compression) $\bar{R}_c$, $\bar{\varpi}_c$ and $\bar{D}$ (shock wave), where $J_c > 1$. Ratios between variables $\hat{f}$ and f on opposite sides of gas-dynamic discontinuities are called dynamic compatibility conditions (DCC) (Uskov, 1980). DCC at stationary discontinuities is a balance of specific flows of (Uskov, 1983)

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- substances

$$\left[\rho\upsilon_n\right] = \hat{\rho}\hat{\upsilon}_n - \rho\upsilon_n = 0 \tag{1}$$

- normal

$$\left[p + \rho\upsilon_n^2\right] = 0 \tag{2}$$

- and tangential components of the momentum

$$\left[\rho\upsilon_n\upsilon_\tau\right] = 0 \tag{3}$$

- energy

$$\left[\rho\upsilon_n h_0\right] = 0 \tag{4}$$

where υ_n and υ_τ are projections of the velocity vector on the plane of discontinuity. DCC in such form weren't created at instant. Differential equations for density ρ and velocity potential φ, describing the one-dimensional non-stationary motion of inviscid perfect isothermal gas were first introduced in 1788 in the book of J.L. Lagrange (1788), the formula are given as in original:

$$a^2 \ln\frac{\rho}{D} + \frac{\partial\varphi}{\partial t} + \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 = 0\,; \tag{5}$$

$$\frac{\partial\rho}{\partial t} + \rho\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial\rho}{\partial x}\frac{\partial\varphi}{\partial x} = 0\,. \tag{6}$$

Here a is a sonic speed, D – density of initially unperturbed medium. For connection between the pressure, density and sonic speed, Lagrange used a ratio, previously proposed by Newton:

$$p = a^2\rho\,. \tag{7}$$

Literature review

The derivative $\frac{\partial\varphi}{\partial x}$ is the gas velocity. In 1808, S.D. Poisson (1808) obtained the expression for it in the form of a plane wave:

$$\frac{\partial\varphi}{\partial x} = F\left[x + t\left(a - \frac{\partial\varphi}{\partial x}\right)\right], \tag{8}$$

where F - an arbitrary function determined by the initial and/or boundary conditions.

In 1848, G.G. Stokes drew attention to the fact that the solutions of these equations stay continuous only for a limited period of time. It's interesting that the problem of studying the motion of an ideal gas containing discontinuities was considered in exactly this form in the late 20th century by the scientific school of Soviet mathematician V.I. Arnold (1988). Equation (7) describes the velocity field of particles freely moving in a straight line. Law of particle free motion is given by x = φ (t) = x0 + ut, where u is the velocity of the particle. The function φ satisfies the Newton equation. By definition, dφ / dt = u (t, φ). Differentiating the last relation in t, we acquire the equation, which received the name “Euler equation”

$$u\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}=0\,. \tag{9}$$

Thus, the description of the motion using the Euler equations for the field of gas-dynamic variables and using Newton's equations for particles are equivalent. It is known that quasi-linear differential equations in partial derivatives are solved by building characteristics. Characteristics of the Euler equations are equivalent to Newton's law for a moving particle (Arnold, 1992) and thus the problem of wave propagation can be solved by building characteristics, along which the material particles move. Figure 1 shows how the Euler equation is solved with characteristics.

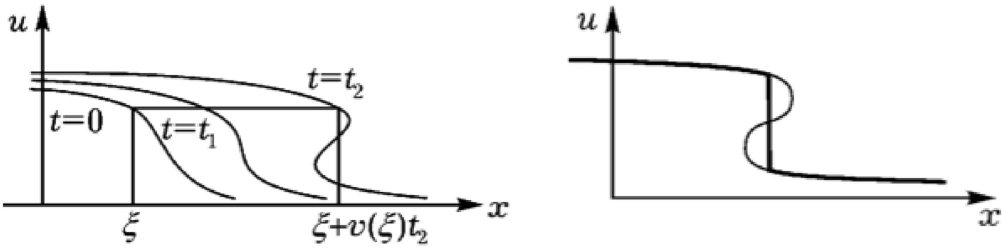

Figure 1. The solution of the Euler equation with characteristics

On the plane y-x the initial function y = u0(x)t = 0 is given. Characteristic equations are t '= 1, y' = 0, x'= y. If the horizontal lines are drawn from this curve, the particles along each of them will move with their constant velocity. Then at some point of time t = t1, t2 ... tn form of the velocity distribution u(x) will change (Figure 2).

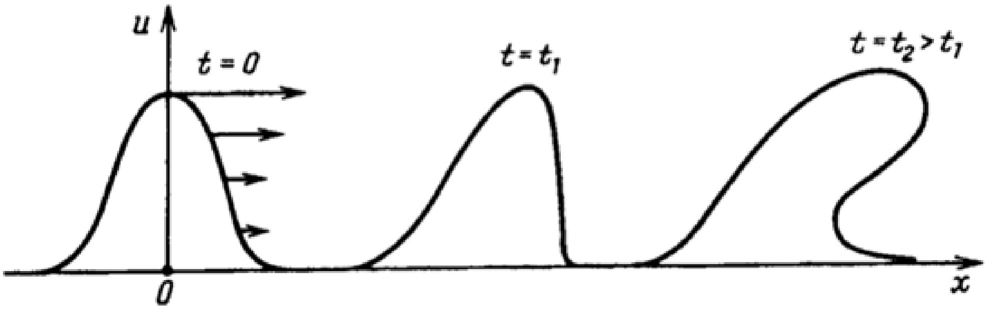

Figure 2. The appearance of ambiguity in the solution

At some point in time (t2 at Figure 2), the mapping u(x) ceases to be the graph of the function, i.e. there are the values of x, which corresponds to more than one u value. In this area, the physical condition of the particles interaction absence means they pass through each other, which is unphysically. It is necessary to introduce some model of their interaction. For example, the model of the universe formation, proposed by Ya.B. Zeldovich (1970) takes into account the expansion of the universe and the gravitational interaction. Adding such conditions creates the features in solution, i.e. areas where the concentration of particles (galaxies for Zeldovich) is maximal. Such areas (set of critical values) are called caustics (Figure 3).

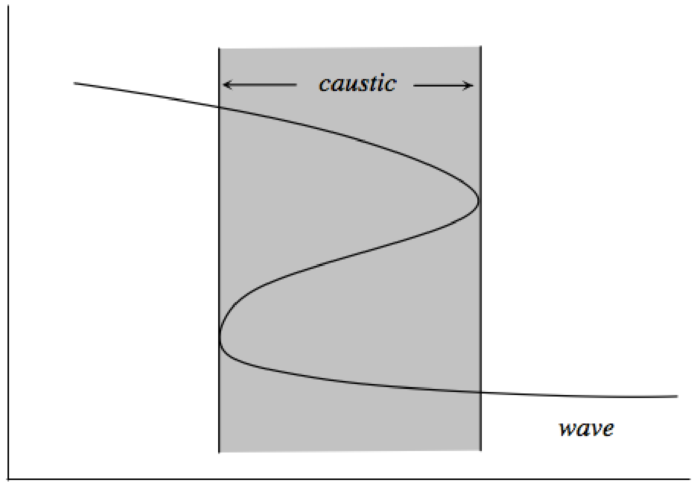

Figure 3. The caustic of a wave

Once established, the caustic can transform, decompose forming new features, but cannot disappear. This model describes the formation of a non-uniform (cellular) structure of the universe from the original random density fluctuations of matter and energy. In our example of the ideal gas supersonic flow it is required to introduce the model of inelastic particles collision. Then, in the location of this collision a shock wave - the discontinuity of particle motion parameters will appear (Figure - 1 right). The discontinuities appear in the solution because Euler field equation ceases to uniquely describe the distribution of gas-dynamic variables. In the work, mentioned above, Stockes firstly introduced the concept of discontinuity in the area of continuous medium and received two conditions for density ρ and gas velocity u on the sides of the discontinuity resulting from the laws of conservation of mass and momentum:

$$\rho_1 u_1 - \rho_2 u_2 = (\rho_1 - \rho_2)\vec{V} \; ; \tag{10}$$

$$(\rho_1 u_1 - \rho_2 u_2)\vec{V} - (\rho_1 u_1{}^2 - \rho_2 u_2{}^2) = a^2(\rho_1 - \rho_2) \, . \tag{11}$$

Here $\vec{V}$ is the discontinuity propagation speed, the index "2" indicates the parameters behind it, the index "1" - the parameters before it (Figure - 5).

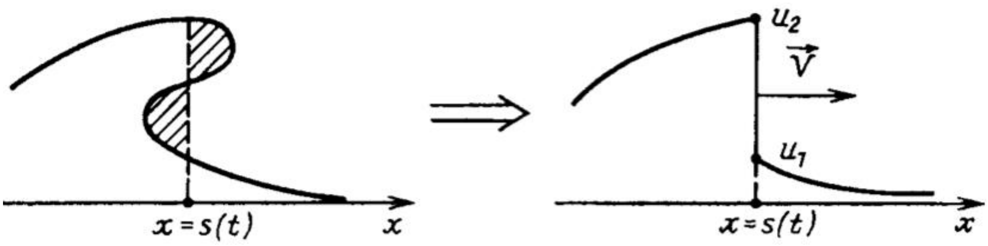

Figure 4. The gas-dynamic discontinuity

Such discontinuities in which the gas-dynamic parameters change abruptly were later called strong. G. G. Stokes said that the abrupt change of parameters on the discontinuities is the result of neglecting the viscosity and thermal conductivity of the medium.

The simplest model of particles inelastic collisions is the Burgers equation (Karman & Burgers, 1939), which describes a gas-dynamic field in smooth regions of space, and the interaction of the particles within the gas shock wave

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}. \tag{12}$$

At low viscosity of ε it approaches the Euler equation in the areas of smooth change of parameters (Figure - 5). Right and left of the shock wave the flow is described by the Euler equations and inside the shock wave (gas-dynamic discontinuity) - by an equation similar to the equation of heat conduction.

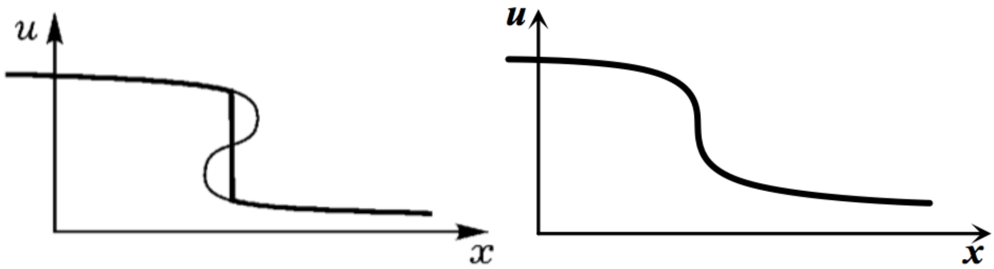

Figure 5. Replacing the Euler equations (left) with Burgers equation (right)

S. Earnshaw (1858, 1860) considered one-dimensional unsteady gas flows, both isothermal and adiabatic. He obtained solutions in the form of a plane wave, which over time can create areas of abrupt parameters change, which he, just like Stokes, called shocks. The speed of disturbances propagation - **the sonic speed** plays an important role in the analysis of gas-dynamic discontinuities and supersonic gas motions. Earnshaw introduces the following relation between the pressure, density and adiabatic sonic speed:

$$a = \sqrt{k\mu}\ , \mu = p / \rho \tag{13}$$

where k - ratio between specific heat capacity of gas at constant pressure to specific heat capacity at constant volume. Today γ is considered a standard notation.

The main purpose of the article is to show the formation of the basic concepts of shock waves theory, with an example of the most significant scientific works (Riemann, 1860; Rankine, 1869, 1870). The history of studying dynamic compatibility conditions for gas-dynamic discontinuities, which determines the ratio between the values of gas-dynamic variables before the discontinuity and right behind it, is reviewed in the article.

Methods

The paper is used analysis of leading scientists from both domestic and foreign schools, who were engaged in issues of this study. The paper is widely used comparative analysis.

Results

Conditions, formulated by Stokes (10-11) were not enough to determine the two unknown parameters of the flow behind the discontinuity and the propagation speed of discontinuity itself. The first attempt to close the equations system written by Stokes was published in 1860 in the work of B. Riemann (1860). In this paper, the author suggests that when going through the normal discontinuity the entropy is constant, and he supplemented the system (10-11) with third equation. In the same time Riemann could not explain the energy change when passing through the discontinuity arising under this assumption.

Independent of the Riemann Rankin in 1869-1870 years (Rankine, 1867, 1870) obtained the third equation, supplementing the system (10-11), in a different form. He discovered the relationship between the parameters on both sides of the shock wave considering the ever-changing state of medium within it, in which the equilibrium heat exchange occurs. The total amount of the heat obtained by the medium must be zero. Using the ratio of equilibrium thermodynamics and the Stokes formula, Rankin obtained the expression for velocity of the normal discontinuity propagation in a stationary medium a (not to be confused with the sound velocity a) and flow velocity u in terms of the known pressure in front of the discontinuity P and behind it p, as well as in terms of known the specific volume S before discontinuity for a perfect gas (notation as in the original):

$$a^2 = S\left\{(\gamma+1)\frac{p}{2}+(\gamma-1)\frac{P}{2}\right\}. \qquad (14)$$

The most important Rankin's result is the statement that normal discontinuities always propagates at supersonic speeds relatively to motionless medium, while relatively to the medium behind the discontinuity their speed of propagation is always subsonic. A method for producing a DCC at the shock wave applied by Rankin leads to the fulfillment of all conservation laws, but it only takes into account the thermal conductivity of the gas, neglecting its viscosity, which is not really reasonable because viscosity and thermal conductivity are interrelated. Hugoniot obtained the conditions on the normal discontinuity more strictly than Rankin, as a consequence of the law of energy conservation, bypassing the consideration of gas state "inside" of the shock wave (Hugoniot, 1889). This condition coincides with the one previously obtained by Rankin, but for obtaining it Hugoniot didn't require any additional assumptions.

Today DDC on stationary discontinuities represent balances of specific flows of (Uskov, 1983):

- substances

$$\left[\rho\upsilon_n\right] = \hat{\rho}\hat{\upsilon}_n - \rho\upsilon_n = 0 \qquad (15)$$

- normal

$$\left[p+\rho\upsilon_n^2\right] = 0 \qquad (16)$$

- And tangential components of the momentum

$$\left[\rho\upsilon_n\upsilon_\tau\right] = 0 \qquad (17)$$

- Energy component

$$\left[\rho\upsilon_n h_0\right] = 0 \qquad (18)$$

Where υ_n and υ_τ are projection of the velocity vector on discontinuity plane. As follows from (15-18) there are 2 types of discontinuities: tangential (τ, where $\upsilon_n = 0$) and normal (shock wave) through which the gas flows. From (16) it is seen that on both sides of τ the static pressures are the same, and from (17) -

that the tangential component may be different i.e. τ are the slip lines. The density, temperature, total enthalpy and entropies of the flows, separated by tangential discontinuities can be different.

The Laplass-Poisson adiabat (isentrope) is easily obtainable from the given system.

$$JE^{\gamma}=1 \tag{19}$$

and so is Rankine-Hugoniot (shock adiabat)

$$E=\frac{1+\varepsilon J}{J+\varepsilon} \tag{20}$$

where $\varepsilon=(\gamma-1)/(\gamma+1)$, γ is the adiabatic index, $E=\rho/\hat{\rho}$, $J=\hat{p}/p$ is the intensity of the shock-wave compression (J> 1) or rarefaction (J <1) process. Isentropic curve (19) is valid for simple compression (J> 1) and waves (J <1) waves, stationary (Prandtl-Meyer waves) or travelling (Riemann waves) ones. Shock adiabat (20) appeared in the simulation of shocks and shock waves by the surfaces of discontinuity. Mach numbers on opposite sides of a wave or discontinuity related by the formula

$$\frac{\mu}{\hat{\mu}}=EJ \tag{21}$$

where $\mu=(1+\varepsilon(M^2-1)$ and $\hat{\mu}=1+\varepsilon\left(\hat{M}^2-1\right)$. Depending on which formula for adiabatic is used (the shock adiabatic formula (16) or isentropic one (15)), the formula (17) makes it possible to determine the Mach number of the rarefaction/compression waves $\bar{\omega}$ and shock waves $\bar{\sigma}$.

In the direct shock wave the ratios $\hat{f}$ and f are set from the system (11-14), in which velocity D is speed of the shock wave, moving along the initial stream, which have the velocity U (Uskov, 2000)

$$\bar{\upsilon}_n=|U-D|\geq a \tag{22}$$

which leads the DDC system to a form of DDC - D:

$$[pu]=[\rho]D; \tag{23}$$

$$\left[p+\rho u^2\right]=[\rho];\ D=[\rho u]D; \tag{24}$$

$$[h_0]=[u]D. \tag{25}$$

When D = 0, the system (22 - 25) describes the DDC on the direct shock wave. (22) shows that for D = U ($\upsilon_n=0$) value $\hat{U}=U$, i.e. $[u]=0$ and there is a surface of variables discontinuity through which the gas does not flow. This discontinuity is called a contact ($\bar{K}$). It moves at gas velocity $\hat{U}=U=D$ and divides the flows shares with different thermodynamic variables (except for static pressures $\hat{P}=P$ as on a tangential discontinuity). By the equations of Clapeyron

$$(\ p=\rho RT=\hat{\rho}\hat{R}\hat{T}\ ) \tag{26}$$

for a perfect gas ($R=\hat{R}$) at $\bar{K}$ these equalities are true:

$$\hat{\rho}\hat{T} = \rho T \ , \frac{\hat{M}}{M} = \frac{a}{\hat{a}} - \sqrt{\frac{T}{\hat{T}}} . \qquad (27)$$

Thus, the contact discontinuity is a special surface that separates the gases with different thermodynamic parameters (except the pressure) (Uskov & Mostovykh, 2010).

From DCC-D (25) it is also clear that, in contrast to shock, on the shock waves exists the discontinuity of total enthalpy as well. The straight shocks themselves are special cases of the standing shock waves (D = 0) in a supersonic gas flow.

Shock polar, heart-shaped curves

A detailed analysis of the gas-dynamic waves (isentropic expansion and compression waves) and oblique shocks arising in the plane, stationary, inviscid, non-conducting perfect gas, was published in 1908 by T. Meyer. In the same paper the parameters of oblique shock wave, formed during the flow around plane acute angle are defined. This is an important task for the practice, because the flow around inclined barriers is one of the most common causes of a shock wave in the gas stream. Starting with this work of Meyer, the intensity of a shock wave (the ratio of static pressures $J = p_2 / p_1$ on its sides) has been considered the main parameter, characterizing. In their modern form DCC on shock waves were formulated by V. N. Uskov in 1980. Later they were developed for the case of one-dimensional traveling waves and oblique shock waves (Uskov, Tao & Omelchenko, 2002). In these works, the convenient formulas for calculating the parameters of oblique shocks and oblique shock waves are given. In particular, for the intensity of oblique shocks ($\bar{\sigma}$).

$$J_\sigma = (1+\varepsilon)\left(\frac{\upsilon}{a}\right)^2 \cdot \sin^2\sigma - \varepsilon = (1+\varepsilon)M^2 \sin^2\sigma - \varepsilon \qquad (28)$$

Here σ - the angle of a velocity vector to the plane of a shock, which can range in the limits $\alpha \le \sigma \le 90°$, where $\sin\alpha = 1/M$ is a Mach angle at which the shock degenerates into a Mach line ($J_\sigma = 1$).Values of J are as well determined by other gas-dynamic variables behind the discontinuities: the density using the shock adiabatic, temperature ($\hat{T}/T = EJ$), sonic speed ($\hat{a}/a = \sqrt{EJ}$). Angle of flow rotation at the shocks is also determined by the intensities J_σ and J_m

$$tg\beta = ctg\sigma \frac{(1-\varepsilon)(J-1)}{J_m + \varepsilon - (1-\varepsilon)(J-1)} . \qquad (29)$$

Here $ctg^2\sigma = (J_m - 1)/(J+\varepsilon) = (E - E_m)/(E_m - \varepsilon)$. In the coordinates $\{\Lambda \equiv \ln J, \beta\}$ the formula (28.29) describes a family of curves (Figure 6), nicknamed for their distinctive appearance the heart-shaped curves. Their other name is the shock polar. Studies of heart-shaped curves, conducted by V.N. Uskov allowed determining their important properties: the presence of the envelope, of the limiting angles of flow deflection at the discontinuities, of the points corresponding to discontinuities, the Mach numbers behind which are equal to one. It can be noted that the presence of the envelope is important in problems of supersonic aerodynamics (Uskov & Chernyshov, 2014) as the pressure corresponds to extrema on the sides of the body of flying a predetermined attack angle, but with a variable velocity.

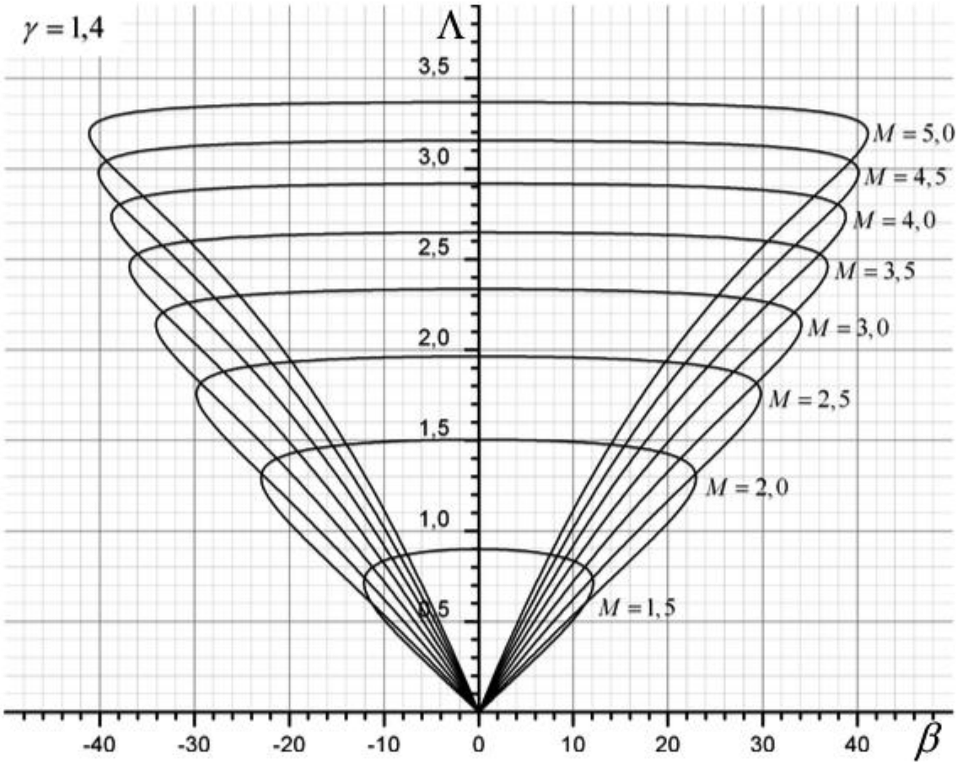

Figure 6. Shock polars

Discussion

The results for the optimal (in the sense of reaching the extremum of some parameter) one-dimensional traveling (Uskov, 2000) and two-dimensional oblique shock waves (Uskov & Mostovykh, 2006) are obtained. Relations describing the heart-shaped curves have long been known, but their use is still often causes difficulties because of the certain computing specifics and the need for selection from a variety of formal roots. The first quantitative experimental results that could be compared with the theory were obtained by P. Vieille (1899). He measured the propagation velocity of the shock wave in the tube after the break of membrane (the prototype of the modern shock tube). A. Stodola (1903) studied the flow in the nozzle on the mode with initial shock wave inside it. These studies provide experimental confirmation for the theory of Stokes, Riemann, Rankine and Hugoniot for a single discontinuity. But in the shock-wave processes may involve not only single waves and discontinuities, but also shock-wave structures (SWS).

Conclusion

This survey provides links to the most important and landmark scientific papers devoted to finding relations on shock, shock waves, simple (isentropic) compression and rarefaction waves. The concept of a gas-dynamic discontinuity as the surface on which the gas-dynamic variables are discontinuous is introduced. The connection with geometric theory of equations in partial differentials is shown. The concept of the shock polar is introduced, the most important works, which studied the properties of shock polars are provided. This article can serve as a brief introduction to the theory of gas-dynamic discontinuities. It sets out the modern idea of a gas-dynamic discontinuity as the features arising in the solution of hyperbolic partial differential equations.

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Disclosure statement

No potential conflict of interest was reported by the authors.

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