Fostering Young Children's Spatial Structuring Ability

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Insight into spatial structures (e.g., dice dot configurations or double structures) is important for learning numerical procedures such as determining, comparing and operating with quantities. Using design research, a hypothetical learning trajectory was developed and an instruction experiment was performed to gain a better understanding of how young children's (aged 4-6 years) spatial structuring ability may be fostered. In this paper we highlight the role of an overarching context in influencing the effectiveness of the instructional setting. The context that was designed for this instruction experiment created opportunities for the children and teacher to focus on spatial structuring in a sequence of instruction activities. The analyses suggest that children benefited from having participated in the instruction activities. In particular, the overarching context helped them to gain awareness of spatial structures and to learn to use spatial structuring strategies rather than unitary counting procedures. This emphasizes the importance of acknowledging spatial structure in early educational practice for cultivating young children's mathematical development.

Keywords: spatial structuring, kindergarten, number sense, design research, context

When asked to determine the quantity of a randomly arranged collection of objects, young children initially tend to count each object unitarily. As the set grows, this procedure eventually confronts them with the difficulties of keeping track of count, and with the time-consuming process that accompanies the counting of large quantities. This calls for ways to physically or mentally rearrange the objects so that the counting procedure may be shortened. In fact, research has shown that children who focus on non-mathematical features and who continue to prefer to count objects unitarily without using any form of structure, may be prone to experiencing delays in their mathematical development (e.g., Butterworth, 1999; Mulligan, Prescott, & Mitchelmore, 2004).

The present research intends to contribute to better understanding of the role of spatial structuring ability for fostering the early mathematical development of young children (Van Nes, 2009). To this end, our aim is to design an instructional setting that fosters this development and supports children in learning to use spatial structures for shortening numerical procedures such as determining, comparing and operating with quantities. For instance, children don't need to count the dots in the dice structure for 6, but learn to recognize two rows of three (:::) or four and two (:: and :). Through exploring and comparing structures, as represented by objects like dice and egg cartons, children can come to recognize and use these underlying structures. We propose that such insight can help to establish and secure children's awareness of spatial structures, to support children's ability to recognize and manipulate such structures in various contexts, and to use spatial structuring to shorten numerical procedures. This research can highlight the need for instruction that promotes spatial structuring strategies rather than unitary counting procedures (Clements, 1999; Clements & Sarama, 2007). As such, the general research question is posed as follows:

- What characterizes an instructional setting that can support children in learning to make use of spatial structures to shorten and simplify numerical procedures?

In this paper, we focus on one particularly important characteristic of an effective instructional setting, namely the context that embeds the sequence of instructional activities.

Theoretical Background

Spatial Structuring and Numerical Insight

We make use of Battista and Clements' (1996) definition to define spatial structuring as

> ... the mental operation of constructing an organization or form for an object or set of objects. Spatially structuring an object determines its nature or shape by identifying its spatial components, combining components into spatial composites, and establishing interrelationships between and among components and composites. (p. 503)

Mulligan, Mitchelmore, and Prescott (2006a) found that children with a more sophisticated awareness of patterns and structures excelled in mathematical thinking and reasoning compared to their peers and vice versa. Although the correlations could not reveal causal effects, the researchers suggested that young children are capable of understanding more than just unitary counting and additive structures. Mulligan, Prescott, Papic, and Mitchelmore (2006b) also found that young (5-12 years), low-achieving students can be taught to seek and recognize mathematical structure and that this can lead to an improvement in their overall mathematics achievement. They concluded that "the development of pattern and structure is generic to a well-connected conceptual framework in mathematics" (p.214), and that instruction in mathematical patterns and structures could stimulate children's learning and understanding of mathematical concepts and procedures. Indeed, Battista and Clements (1996) and Battista, Clements, Arnoff, Battista and Van Auken Borrow (1998) found that students' spatial structuring abilities provide the necessary input and organization for the numerical procedures that third, fourth, and fifth grade students used to count an array of squares.

The research above suggests that children's ability to spatially structure is essential for the development of insight into numerical relations. This insight involves the structuring (e.g., (de)composing) of quantities (e.g., understanding six to be three and three, but also five and one, or four and two, Hunting, 2003; Steffe, Cobb & Von Glasersfeld, 1988; Van den Heuvel-Panhuizen, 2001; Van Eerde, 1996; Van Nes & De Lange, 2007), which, in turn, is essential for the development of higher-order mathematical abilities such as counting and grouping (Van Eerde, 1996; Van den Heuvel-Panhuizen, 2001). Spatial structuring also underlies part-whole knowledge in addition, multiplication and division (e.g., 8 + 6 = 14 because 5 + 5 = 10 and 3 + 1 = 4 so 10 + 4 = 14), the ability to compare a number of objects (i.e., one dot in every one of four corners is less than the same configuration with a dot in the centre, Clements, 1999), to extend a pattern (i.e., repeating the structure, Papic & Mulligan, 2005, 2007), and to build a construction of blocks (i.e., relating characteristics and orientation of the constituent shapes and figures to each other, Battista et al., 1998; Van den Heuvel-Panhuizen & Buijs, 2005).

Towards an Instruction Theory on Spatial Structuring

The ongoing research on the development of young children's structuring and patterning ability calls for more insight into the characterization of the developmental trajectory, as well as into the influences that the instructional setting may have on children's development of spatial structuring ability. This implies that research must focus on designing interventions that foster children's understanding of number sense and mathematical procedures starting as early as in a kindergarten[1] setting. The principles of Realistic Mathematics Education (RME; Freudenthal, 1973, 1991; Gravemeijer, 1994; Treffers, 1987) offer guidelines for designing, conducting, and interpreting such research.

The term *realistic* in Realistic Mathematics Education implies that the problem situation is set in a context that gives a problem meaning and that brings forward the mathematics that "begs to be organized." At an initial level of learning, "realistic" does not have to be true in real life (e.g., it may be a context with fairy tale characters or a context in a mathematical setting), as long as it is "experientially real" to the student, so that it gives meaning to the student's mathematical activity. Such a context can be motivating, but it is especially important that it acts as a model for stimulating personal strategies that can be used as building blocks for the mathematics that is the focus of the discussion.

The intervention is aimed not only at cultivating young children's spatial structuring ability, but also at contributing to an understanding of *why* a particular instructional setting may or may not support young children's learning. This required cumulative cyclic, classroom-based design research (Gravemeijer & Cobb, 2006). Design research involves formulating, testing and refining a hypothetical learning trajectory (HLT) and a corresponding sequence of instructional activities for the teaching experiments. The HLT included testable conjectures that outlined how the intervention was expected to influence the children's learning processes. These conjectures provided for a connection to an instruction theory about how young children can be supported in the development of their spatial structuring ability. The teaching experiments resulted in an empirically supported contribution to this instruction theory about the process of learning. This contribution includes a learning trajectory that is based on mathematical, psychological, and didactical insights about how the children are expected to progress towards an aspired level of reasoning (Gravemeijer, 1994; Gravemeijer & Cobb, 2006). Such a progression should take into account both the cognitive development of the individual students, as well as the social context (i.e. people, classroom culture and type of instruction) in which the teaching experiments are to take place (Cobb & Yackel, 1996; Gravemeijer & Cobb, 2006).

In practice, such an instruction theory encompasses an instructional sequence, as well as a description of the coinciding learning processes, the classroom culture, and the proactive role of the teacher. Hence, by implementing a sequence of instructional activities in a classroom setting, we expected to create an ecologically valid instructional setting in which the children

[1] In the Netherlands, K1 with four to five-year olds and K2 with five- to six-year olds form a part of primary school where children do not yet receive formal education.

could interact with each other and with the teacher and learn about how to make use of spatial structures to simplify numerical procedures (Cobb & Yackel, 1996).

Design and Research Methodology

The first teaching experiment with the sequence of instructional activities resulted in data on the learning processes of the children and on the effects of the activities. The data, in turn, provided the input for a second run with a set of revised activities. This iterative procedure contributed to gain further insight into how the children were learning to make use of spatial structures as a means to shorten their numerical procedures. In this section we present the participants and setting of the research and explain the procedure for both rounds of teaching experiments.

Participants and Setting

The study was conducted in a kindergarten classroom at a local elementary school. The children at the school had mixed social and cultural backgrounds. The kindergarten class that participated in the experiment was a combined grade 1 and grade 2, for a total of 21 children that ranged in age from four to six years. Pre- and post-interviews were conducted with the children who participated in the experiment (i.e., the intervention group, “IG”) as well as with a comparable kindergarten class (i.e. the non-intervention group, “NG”) of 17 children who only participated in the pre- and post-interviews and not in the experiment. Although the NG was not strictly a control group, it was included in the research to enrich the analyses on the IG children’s post- compared to pre-interview performances by looking into whether these outcomes show any differences with the outcomes of the NG.

Procedure for Round 1

The IG was taught by two teachers and these teachers performed the instructional activities with the class while the researcher observed and asked the children additional questions, helped to coordinate the activity, took field notes, videotaped the lesson and made last-minute revisions to the activity. The researcher discussed the activity with the teacher in the half hour before the lesson to prepare her for teaching the class on her own. The data consisted of video recordings of each of the instructional activities, the questionnaires that the teachers completed for debriefing, the log that was written about what happened during the activity, and additional notes from discussing the activity with the teacher before and after the session.

The instructional sequence that was tried out in Round 1, consisted of six instructional activities that were inspired by literature, consultations with experts and classroom experiences. The focus of the activities progressed from a predominantly spatial focus (decomposing geometric shapes and patterning), to a spatial structuring focus (constructing with blocks and a type of Bingo game with dot configurations and double-structures), to finally a focus on number sense (structuring chips to keep count). The underlying conjecture was that children’s understanding of spatial structures can support them in recognizing, making use of, and applying spatial structures to shorten certain numerical procedures. As such, each activity was intended to draw on the insights that are the topic of discussion in the previous activity. Each instructional activity started as a classroom discussion that was guided by the teacher. Then the researcher took five children aside (the focus group), for more in-depth discussions and detailed observations of their approaches to the activity.

After performing the instructional sequence with the IG, the children's performance and the instructional setting was analyzed qualitatively. We focus in this paper on the observation that neither the children nor the teacher were explicitly or implicitly making reference to previous activities, while it was expected that they would make use of previously gained insights to approach the present activity. This suggests that a context was missing that could link the activities together. This inspired revisions to the instructional sequence, which were tried out in Round 2 with the IG as explained below.

Procedure for Round 2

Considering our observations from Round 1, we decided that an overarching context could motivate the children and help them understand the essence of the activity and the use of tools that represent spatial structures in light of the previous activities and insights. An appealing context can also contribute to creating a shared vocabulary about spatial structuring. The teacher can, for example, guide the children towards a more spatial structuring approach by asking the children whether they remember how they used spatial structures in a previous activity and whether that strategy could help them in the present activity. As such, in Round 2 we introduced "Ant and its Tool Box" (in Dutch "Miertje Maniertje" and the "ManiertjesDoos" rhyme and the name sounds appealing to the children).

Ant became an important figure in the experiment because it excited the children and its Tool Box played an important role in bridging each of the five activities. While in Round 1, each of the instructional activities had their own attractive contexts, in Round 2 Ant's Tool Box became an overarching context. The significance of this is that it supported the children in making practical and theoretical connections between the activities themselves and between the insights that the children may have gained during the previous activities.

Figure 1. Ant and its Tool Box.

We conjectured that a strong introduction would spur the children's curiosity. Hence, to prepare for the first lesson, the box was placed in the middle of the classroom and the ant was hidden on a bookshelf. Several pieces of paper were spread out on the floor, leading from the entrance of the classroom to the box and beyond the box to the bookcase. On the papers were drawn two rows of three dots. These represented the footprints of the ant which it had left behind while carrying the box into the class for the children to find. The reason, then, for choosing an ant as the main character in this context is that an ant has six legs (i.e., a fundamental spatial structure), that the ant's name conveniently relates to the name of the box in Dutch, and, finally, that ants appeal to children's imagination.

The Tool Box contained large cards with finger patterns, two large dice, large cards with playing card configurations, several egg cartons for six eggs and for ten eggs, and a box with several types of patterned bead necklaces. The story is that the Ant had "tools" that it wanted to share with the class because these could help the children to determine a quantity. By first agreeing to call the contents of the box Ant's "useful tools for determining a quantity", the teacher created a shared vocabulary with which she could repeatedly refer to Ant and its tools throughout the rest of the activities. In this way, she could refer to the contents of the box and stimulate the children's spatial structuring approaches to a particular activity. The next section highlights several results that illustrate the influence of the overarching context on children's spatial structuring strategies.

Shortly after Round 2, the IG and NG children performed in a post-interview; the types of spatial structuring strategies that they used to solve the numerical interview tasks were quantitatively and qualitatively compared to the types of strategies they used on the pre-interviews, which were conducted before the experiment. This was to provide more insight into whether and how the instructional sequence influenced the children's development of spatial structuring. The teachers were also interviewed after Round 2 to evaluate how the experiment influenced their perspectives on teaching about spatial structuring and on the role of spatial structure in young children's early mathematical development.

Data Analysis

The data was analyzed qualitatively with the help of the multimedia data analysis program ATLAS.ti. This program provides a format for organizing the raw data into clips that simplify the process of tracing behavioral patterns (Jacobs, Kawanaka & Stigler, 1999). After importing raw data in the form of, for example, a video, screenshots or scans of written work into the program, the researcher can organize the data in ATLAS.ti by segmenting the data into "quotations" (i.e., video clips or "meaningful chunks"; Stigler, Gallimore & Hiebert, 2000).

Through adding comments to quotations, creating codes to label the quotations and linking the appropriate codes to specific quotations, we could make sense of how the children were solving the problems, how they were developing in their understanding, how the researcher, the teachers and the instructional activities had played a role in this development, and how proactive individual and classroom instruction could ultimately support the children's learning. The insights were supplemented with data from the debriefings with the teachers and reflections on the interviews with the children and the classroom activities.

Earlier in the research we had also developed a strategy inventory to gain insight into children's level of spatial structuring ability as they performed the specially developed interview tasks. This strategy inventory was another reliable instrument (with a Cohen's Kappa value of 0.87) for interpreting children's behavioral patterns in the teaching experiment (Van Nes, 2009).

As such, we studied significant episodes in the videos of the instructional activities and noted various underlying behavioral patterns. These meaningful episodes were subsequently summarized into several elements that appear crucial to the design of an effective instructional sequence. Analyses of these elements resulted in an empirically supported contribution to an instruction theory for fostering children's spatial structuring ability. In the next section, we elaborate on the role of the context in the learning trajectory. The learning trajectory itself will be outlined in the discussion.

Results

In the analyses of the Round 2 of the teaching experiment, we were able to cluster observations that concern three areas in which the new overarching context of Ant and its Tool Box appeared to have contributed to the design of an effective instructional setting.

The first area is *children's motivation and their identification with the instructional activities*. Our analyses show how the context sparked the children's interest and motivated them to participate in the activities; the children were excited to discover why Ant had left the Tool Box in the classroom, they were keen to unpack the box, and they started counting the egg cartons on their own initiative. One child recognized, for example, that the number of egg cartons he counted is "how old I am". These kinds of remarks and reasoning created an opportunity for the teacher to start a discussion about what ways, other than unitary counting procedures, there are to, for example, determine a quantity.

Figure 2. The children are excited to explore the contents of the Tool Box

The activity also appealed to children's different levels of learning. This was observed when the teacher asked the children to determine the number of footprints on the papers on the floor, and one child counted the dots unitarily while another child recognized the structure for six as "two rows of three". The discussion that followed encouraged the children to compare their strategies and see what role spatial structuring may play in shortening their counting procedures.

Second, the context played an important role in *connecting the activities and in stimulating children's attention to spatial structures*. In one activity, for example, the children were asked to determine the next layer of blocks in a 3D block construction, based on the structure of the layers at the bottom. The teacher could refer to the overarching context about Ant building its ant hill (i.e., the block construction) and ants marching in a procession (i.e., the previous patterning activity) to encourage the children to try to abstract the structure of the construction in the same way as they had done in the patterning activity. In this way the children understood better that they could make the ant hills taller by studying the structure of the construction. They said that in this way they could "see better how it's put together".

Moreover, at the start of a new activity, the children vividly recalled the context of the preceding activities. They spontaneously talked about how they enjoyed the activity where Ant came to pick flowers. The children also remembered how the "tools" (i.e., the types of spatial structures) in the Tool Box helped them to see, for example, how many of something there are without having to count the objects unitarily. In this way, the children became familiar with the contents of the box and explored how the objects represent types of spatial structures that can support them in the activity.

Finally, the context highlighted the *important role of the teacher in supporting children's learning*. The Tool Box enabled the teacher to make reference to various types of spatial structures and to encourage children to associate unfamiliar arrangements of objects (e.g., flowers arranged in rows) with relatively familiar structures in the box (i.e., dot arrangements on dice). As such, she helped the children to compare various types of structures for one quantity (e.g., finger patterns and dice configurations to represent six) as well as to study how various quantities are represented with one structure (e.g., arrangements of eggs in an egg carton). Moreover, the shared vocabulary that the teacher established was manifested both during and after the experiment. An example of a shared phrase is "easy ways" to determine a quantity. Throughout the teaching experiment, the teacher and children used "three, three", for example, as a shared way to describe the symmetrical structure of six as two rows of three. The significance of this is that, during the interviews that were held with the children individually after the experiment, children tended to use the same vocabulary and to refer to the instructional activity contexts to explain their approaches to the interview tasks. This suggests that the children were able to recognize a spatial structuring opportunity by translating their approach to the instructional activity to the interview setting.

Overall, the qualitative analyses of the instruction experiment after Round 2 reflected benefits of an instructional setting that supports awareness of spatial structuring for fostering young children's insight into numerical relations. In addition, the outcomes of the post-interviews showed, for example, that 18 out of the 21 intervention group children increasingly started referring to spatial structures and discussing the conveniences of spatial structuring procedures over unitary counting, through the use of the shared vocabulary. Moreover, the teachers who participated in the teaching experiment reported that they themselves had gained awareness of spatial structures as well as a greater appreciation for the importance of spatial structuring ability for young children's mathematical development.

Finally, one year after performing the experiment, it was observed that the teachers introduced Ant in their classroom instruction on their own initiative, and that several children spontaneously made reference to Ant in their practice at determining a quantity.

Discussion and Conclusion

In this paper we gave an impression of the role of the overarching context in helping children become more aware of the convenience of spatial structuring for simplifying and shortening mathematical procedures such as determining, comparing and operating with quantities. This research is part of a larger study that suggests that the context of Ant and its Tool Box contributed to the effectiveness of the instructional sequence and illustrate how an overarching context is an important component of an instructional setting that requires attention in the process of designing and revising a HLT for the development of young children's spatial structuring ability (van Nes, 2009).

The analyses of the two rounds of the experiment culminated in characteristics of a learning trajectory which are outlined in Figure 3 (cf. Gravemeijer, Bowers, & Stephan, 2003; Van Nes, 2009). We refer to the first column as Tools to indicate that our aim is for children to experience each activity as a natural follow-up of activities. The children should be able to recognize their earlier structuring activities in the new tool. That is the focus of the classroom discussion. Next, we describe the imagery (or history), which is the type of knowledge and experiences that the lesson builds on. The third column describes the activity that was performed. The last column includes the mathematical issues that should arise during the discussions about the activity. These issues are expected to inspire children towards new levels of understanding, and prepare them for the next activity in the sequence.

As such, the instructional activities in the experiment progressed from the introduction of the context (i.e., the box and its contents), to two activities in which the children had the opportunity to explore the spatial structures of objects in the Tool Box, to two activities in which the children were challenged to use the relatively unfamiliar structures in the activity in the same way as they had used the structures in the box. Finally, in the last activity the children were encouraged to apply spatial structure to relatively larger unstructured configurations of objects as a means to shorten the process of determining and comparing quantities (Van Nes, 2009).

Considering the explorative rather than confirmative nature of this design research, we are careful not to draw definite conclusions about the instructional sequence. Nevertheless, our research provides valuable insights for both scientific and practical purposes. The experiments have, for example, resulted in a sequence of instructional activities embedded in a context that helped those particular children who participated in this research become aware of spatial structures and of the convenience of making use of spatial structures in determining, comparing, and manipulating quantities. This complements Mulligan, Prescott and Mitchelmore's (2004) spatial structuring developmental trajectory and it supplements their research with an instruction theory for fostering children's progression in such a developmental trajectory.

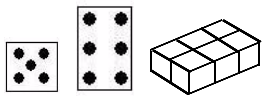
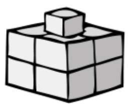
Tool	Imagery	Activity	Mathematical issues
A box containing ordinary objects that represent familiar spatial structures	Experiences in daily-life situations (e.g., playing with egg cartons, dice)	Introducing Ant and the mystery of the Tool Box	Exploring Ant's Tool Box and creating awareness of similar spatial structures
Objects that represent dot configurations (e.g., symmetric and double-structures, five-patterns) 	Experiences in daily-life situations (e.g., playing with egg cartons, dice)	Recognizing and comparing dot configurations	Exploring spatial structures as "tools" for recognizing, determining and comparing quantities (i.e., relating structures to quantities)
Abstracted spatial structures 	Spatial structures in daily- life situations	Recognizing and comparing structured and unstructured objects	Using and comparing structures as "tools" for dealing with quantities (e.g., "seeing" 2 rows of 3 with 1 as 7 in a dot configuration)
Patterning with children and with colors 	Abstracted spatial structures (for (de)composing patterns)	Creating and describing patterns	Abstracting structure from, and applying structure to patterns (e.g., 2-1-2-1 or 3-3-3)
Structured 3-D block constructions 	Patterning for (de)composing 3-D constructions	Building and analyzing 3-D constructions and determining the number of blocks in the construction	Patterning as a "tool" for analyzing 3-D constructions and numerical relations (e.g., layers of 4, 4, and 1 blocks makes 9 blocks)
Spatial structures	Daily life objects, symmetry, patterning, structures and connected quantities	Determining and comparing unstructured quantities	Structures and number relations as "tools" for organizing and comparing quantities

Figure 3. Outline of a learning trajectory for the development of children's spatial structuring ability (adapted from Van Nes, 2009).

Taken together, the children's and teachers' excited and fruitful responses to the activities encourage more systematic investigations into the role of this instructional sequence in supporting children's spatial structuring ability. The instructional sequence of activities could support those particular kindergartners who may already be at risk for developing mathematics learning problems with instruction that is tailored to appeal to their mathematical strengths (e.g., early spatial structuring ability) and interests as a way to approach their relative weaknesses (e.g., problems with counting). At the same time, it offers a framework of reference for planning instruction that can challenge high-achieving children (e.g., associating spatial structure with formal mathematical procedures such as multiplication). As such, the research may contribute to ways of furnishing a supportive instructional setting to cultivate children's mathematical development and offer them a head start in their formal mathematics education.

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