

Didactical Conceptualization of Dynamic Mathematical Approaches: Example Analysis from the InnoMathEd Program

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This paper examines examples of teaching approaches involving the use of dynamic mathematics software. These were nominated as successful approaches by four organizations participating in the InnoMathEd professional development program. The most common type of task reasoning structure was one which required students to quantitatively formulate a mathematical relationship expressed by a visual representation. The examples nominated by most of the organizations reflected a more didactic, guided-discovery orientation grounded in directed action, but those from one organization reflected a more adidactic, problem-solving orientation grounded in constrained solution. Instrumental demands on students varied substantially between examples, calling for very different levels of preparation and guidance. The core idea behind these dynamic approaches was one of manipulating displayed representations so as to highlight associated variation (or non-variation) of properties, and relations between different states or representations. Employing this type of user interaction to support visualization and observation was seen as creating a learning environment that encourages exploration and experimentation through which mathematical properties and relationships can be discovered.

Keywords: Dynamic mathematics software, professional development, computers, problem solving

Sustained professional interest in the potential contribution of digital technologies to mathematics education has not yet been translated into widespread use of such technologies in everyday mathematics teaching. A key intermediary factor is the professional development available to teachers. Gaining insight into the examples of technology use that teachers are exposed to, and the ways in which these are conceptualized didactically, provides a base for reflection on the effectiveness of current professional development provision in this crucial process of intermediation. In this paper, we examine this issue by drawing on the work of a professional development program in which well-established providers of initial and continuing teacher education collaborated in promoting the use of didactical approaches based on the use of dynamic software in school mathematics.

The Innovations in Mathematics Education Project

Innovations in Mathematics Education [InnoMathEd] was a professional development project involving organizations in several European countries (Bianco & Ulm, 2010). The InnoMathEd proposal argued that learning environments based on the use of dynamic mathematics can serve as catalysts for pupils' active, independent and exploratory learning in school mathematics. More specifically, the project intended to focus on the use of a particular type of learning environment to support such learning: the "dynamic worksheet" in which manipulable constructions and accompanying text serve to pose problems or provide explanations.

The InnoMathEd proposal highlighted a number of significant features of the mathematical constructions that dynamic resources make possible.

They allow visualizations that are not possible by means of traditional media for teaching. In contrast to constructions on paper or on the blackboard, constructions with dynamic mathematics can be changed and varied on the screen dynamically. Furthermore integrated computer algebra systems bridge gaps between geometry, algebra and calculus. They allow to measure lengths, angles or the coordinates of points and to use these measurements for further calculations. They offer working with functions and integrating graphs of functions in dynamic constructions. (InnoMathEd, 2008)

Each of the major partner organizations in the project was involved in providing some form of initial and/or in-service teacher education on the design and use of dynamic resources in school mathematics. Our own more modest role was, at an early stage, to identify ideas from published research on dynamic mathematics that highlighted significant implementation issues (to be presented in the next section); and then, at a later stage, to examine what forms of use of dynamic mathematics the major partners had found to be particularly successful (to be presented and analyzed in later sections).

Significant Issues in Implementing the Use of Dynamic Mathematics

Previous research studies have identified a number of significant issues in developing the use of dynamic resources in school mathematics.

In a French project aimed at developing the classroom use of dynamic geometry, teachers devised lesson ‘scenarios’, and refined them through discussion with a wider team of educators and developers associated with the software (Laborde, 2001). In the more conventional scenarios developed by teachers, dynamic geometry provided a convenient parallel to paper and pencil: to produce accurate static figures and generate measurement data; to highlight invariant properties through their visual salience under dragging. Scenarios of these types were often marked by “a restricted and static use of the possibilities of the software,” and “the absence of autonomous experimentation by students” (Laborde, 2001, p.299). This accords with critical review of the dynamic geometry activities proposed in the professional literature which suggests that many of these reduce to a verifying approach in which students are simply expected to vary geometric configurations in order to produce empirical confirmation of already formulated results (Hölzl, 2001). However, the French project did give rise to some more ambitious scenarios, making fuller use of dynamic software to support more interactive tasks, and often depending on mathematically innovative thinking: to pursue qualitatively new types of solution to familiar problems, as with certain aspects of construction; and to pose novel types of problem dependent on the software, such as identifying the properties of a dynamic diagram through dragging it. Even in this very well supported development project, then, the ways in which dynamic geometry was used within classroom practice varied considerably with respect to their degree of mathematical innovation, interactive organization and exploratory orientation.

A British study of the existing use of dynamic geometry in well-regarded school mathematics departments found that a shared rhetoric of employing dynamic geometry to support guided discovery accompanied very different forms of pedagogical thinking and classroom practice (Ruthven, Hennessy & Deaney, 2008). First, teachers took very different views on whether or not students should themselves work with dynamic software: at one extreme, the teacher alone managed use of the software, presenting students with carefully prepared figures; at the other, students themselves were expected to construct and manipulate relatively complex figures. The framework within which teachers evaluated the costs and benefits of student use was strongly influenced by the extent to which they saw such activity as providing students with direct experience of a mathematical reference model for the topic under instruction, and more fundamentally as promoting mathematically disciplined interaction. Likewise, teachers took very different approaches to handling apparent mathematical anomalies of software operation: at one extreme, seeking to ensure that these remained hidden from students; at the other, viewing student exposure to such situations as supporting critical thinking and strengthening mathematical understanding. Teachers' choice of approach depended on whether they saw such anomalies as providing opportunities to develop students' mathematical understanding, in line with a more fundamental pedagogical orientation towards supporting learning through analysis of mathematical discrepancies. In effect, variability in students' exposure to the operation of dynamic geometry was driven by the degree to which teachers saw developing their instrumental knowledge of the software as promoting mathematical learning.

The main thrust of these earlier studies was highlighted at the first InnoMathEd workshop, and the full papers were made available as InnoMathEd resources. They have also provided sensitizing concepts for this analysis of examples of dynamic mathematical approaches developed within the program.

Eliciting Examples of Dynamic Mathematical Approaches

Each of the major partners in the InnoMathEd program was asked to nominate one or two examples of lessons/activities that participating teachers had found particularly successful, and to provide the following information about each example.

- 1) A specification for the lesson/activity written in a way that offers guidance for teachers attempting it for the first time.
- 2) A copy of the dynamic materials to be used in the course of the lesson and of any other supporting materials needed.
- 3) A covering statement, based on discussion with participating teachers, covering the following matters, and any others considered relevant:
 - a. What is it that makes this approach a “dynamic” one?
 - b. What features of this approach lead to it being particularly successful, and how do they contribute to this success?
 - c. What preparation/actions/guidance by the teacher are important to the success of this approach?
 - d. What potential obstacles are there to this approach being successful and how these can be avoided/overcome?

We received responses from four of the participating organizations, providing eight examples of dynamic mathematical approaches judged to be successful (as summarized in Table 1). Each example has been given a reference label and is illustrated in one of Boxes 1 to 8. One important point to note is that all but the last example appear to be at a mathematical level intended to be suitable for use with school students (as well as of value in the professional development of teachers), whereas the last example [D2] appears to be at a higher mathematical level (but intended to provide teachers with a model of the type of pedagogical process that could be used with school students). Further accounts of the work undertaken by these organizations are available in the InnoMathEd end-of-project publication (Bianco & Ulm, 2010): specifically from the Augsburg team (Bianco, 2011; Ulm, 2010); from the Bulgarian team (Chehlarova & Sendova, 2010; Sendova & Chehlarova, 2010); from the Cyprus team (Mousoulides et al., 2010; Sophocleous et al., 2010); and from the South Bohemian team (Binterova & Fuchs, 2010; Pech, 2010).

Table 1

Examples of dynamic mathematical approaches nominated as successful

Participating organization	Title/Mathematical topic	Supporting software	Reference label	Box with illustration
University of Augsburg	Circumference	GeoNext-based applet	A	1
Bulgarian Academy of Sciences	Explorations with configurations of colored unit cubes	Cubix Editor, Cubix Shadows, Cubix	B1	2
	Nets	Origami Nets, Stuffed Toys, and Scissors	B2	3
	There are enough rotational solids to go around	Math Wheel, Potter's Wheel, and Bottle Design	B3	4
University of Cyprus	Area of a circle	EucliDraw Jr	C1	5
	Properties of quadrilaterals	EucliDraw Jr	C2	6
University of South Bohemia	Creating a concept of quadratic function	Cabri, GeoGebra	D1	7
	Heights of the triangle	Cabri, GeoGebra, CoCoA	D2	8

Each submission is illustrated by a box in the Appendix. The wording of these has been taken directly from the original examples, but, since some of the worksheets involved a lengthy series of tasks, they have been shortened to fit this paper. For instance, because D1 involved a total of 31 tasks, we only included the initial ones and then the final one with a view to enabling readers to grasp the intentions of the developers. Full details of these task sequences can be accessed on the InnoMathEd project website.

Analyzing the Nominated Examples of Dynamic Mathematical Approaches

After reviewing the submitted texts and examples, we identified three guiding questions for our analysis. First, what mathematical knowledge did the example seek to develop in students? Second, what was the basic reasoning structure of the task, paying particular attention to the mode(s) of reasoning implied and the associated degrees of freedom? Third, what were the key instrumented techniques required of students in undertaking the task, in the sense of the mathematico-technical competences of tool use? In identifying these features, we sought to be guided as far as possible by the original presentation of the example, but we recognize that because our framework for analysis goes beyond the terms in which these presentations were couched, our conclusions inevitably involve a degree of further interpretation and inference. In addition to the three framework questions, we also analyzed project participants' descriptions of the didactical thinking behind the implemented dynamic features, as given in response to part 3 of our request (with subheadings a to d). The mode of analysis we used was organized around focused coding and recoding of the submitted materials through a recursive process of constant comparison (Strauss & Corbin, 1990; 1998).

Mathematical knowledge and reasoning structures. We examined the mathematical knowledge targeted by these examples further with a view to locating it within a broader typology (as shown in Table 2). All but one of the examples appear to involve some element of spatial visualization [Type-SV, as shown within A, B1, B2, B3, C1, D1, D2]. All but one incorporate some element of verbal-symbolic formulation of quantitative relations [Type-QF as shown within A, B1, B3, C1, C2, D1, D2]. Notably too, the great majority adopt a sequencing in which SV is followed by QF [A, B1, B3, C1, D1, D2]. Thus, the most common type of mathematical goal within the examples is to quantitatively formulate the mathematical relationship expressed by a visual representation. For instance, in example A, students take measurements from circles of different sizes and make conjectures about the relationship between radius and circumference. Less commonly, examples call for coordination of representations [Type-CR, as shown within B2, D1]: in example B2, for instance, polyhedral shapes are represented both as 2D nets and as 3D solids.

We also examined the basic reasoning structure of the tasks (as shown in Table 2) with an eye to whether this involved directed action (in the sense of the student being told what mathematical procedures to carry out or mathematical features to observe) [exemplified by A, C1, C2 (first part), D1, D2] or constrained solution (in the sense of the student being required to find processes or formulate features subject to given constraints) [exemplified by B1, B2, B3, C2 (second part)]. It is clear that the examples nominated by three of the participating organizations [A, C, D] primarily reflect a guided-discovery orientation grounded in directed action, whereas those from the fourth [B] primarily reflect a problem-solving orientation grounded in constrained solution.

Table 2
Mathematical knowledge and reasoning structure

<i>Example</i>	<i>Mathematical knowledge</i>	<i>Basic reasoning structure of task(s)</i>
A	Visualization of the linear character of the circumference of a circle [SV] Formulation of the relation between the lengths of diameter and circumference of a circle [QF]	Directed data pattern generation and formulation
B1	Visualization of the volume of a rectangular cuboid and associated decompositions and recompositions [SV] Formulation of associated spatial patterns and quantitative relations [QF]	Constrained figure construction
B2	Coordinated visualization of polyhedral shapes and their nets [SV, CR]	Constrained problem solving
B3	Visualization of solids of rotation and associated decompositions and recompositions [SV] Formulation of associated quantitative relations for surface area and volume [QF]	Constrained problem solving
C1	Visualization of decomposition of circle into sectors and re-composition into near-parallelogram [SV] Formulation of relation between area of circle and lengths of its radius and circumference [QF]	Directed figure construction Directed data pattern generation and formulation
C2	Formulation of metric properties of standard quadrilateral types [QF] Identification of quadrilateral type satisfying multiple metric properties	Directed figure construction Directed property recognition Constrained figure formulation and trial construction Constrained property formulation and trial measurement
D1	Visualization of quadratic functions from a geometric construction (estimate the locus of a point) [SV] Obtaining the equation of the locus curve by a built-in functionality of the software and connecting it with algebraic representations [QF] Changing parameters in the equation and investigating their role in the graphing of the curve [CR]	Directed figure construction Directed visual pattern recognition Directed automated property formulation Directed property recognition
D2	Visualization of triangle and its altitudes, construction of triangle and its altitudes [SV] Verification with DGS of concurrency of altitudes Using CAS to generate proof [QF]	Directed figure construction Directed property verification and validation

Software types and instrumented techniques. We assessed the instrumental demands of each example (as shown in Table 3) by identifying (as far as available information permitted) the key instrumented techniques required by students. We found that Example A differs from the others in two significant respects. First, it minimizes the instrumented technique required of students through presenting them with a prepared dynamic figure; nevertheless, students still need to make mathematical sense of what is taking place when they drag a point, calling for them to mentally reconstruct the figure. Second, this example incorporates parallel tasks, one using idealized figures in the dynamic environment [Exercise 1], the other using real objects and hand measuring tools [Exercise 2]; however, what remains implicit is the process by which the mathematical parallels between these two situations are constructed. All the other examples call for more extensive instrumented technique on the part of students: generally, this is specifically prompted within the worksheets. Equally, the other examples do not use other tool systems in parallel (notably pencil-and-paper or manipulatives). The dominant model for task design, then, is one in which established tools are replaced, rather than complemented, by dynamic software.

In all of the nominated examples, use of the software is intended to support visualization of mathematical objects. Sometimes the software is designed for a very specific topic within mathematics [B1, B2, B3] while at other times it is a more generic mathematical tool [A, C1, C2, D1, D2]. For instance, the Cubix Editor in B1 is designed specifically to visualize constructions with unit cubes, and to allow coloring of these objects. In contrast, examples C and D use more generic and flexible software such as EuclidDraw, Cabri, and GeoGebra capable of more wide-ranging and versatile constructions.

The instrumented techniques required in the examples appear to make three levels of demand, governed by the closeness of the relationship between the task(s) posed and the supporting software artifacts. The link is tightest and instrumental demands least where a (Type-SA) specific applet is customized to a particular task [A], intermediate where a (Type-TE) tailored environment is designed to support work around a particular topical theme [B1, B2, B3], and the link loosest and instrumental demands greatest where (Type-GS) generic software is used (reflected in the greater degree of instrumental guidance and direction provided) [C1, C2, D1, D2]. At one extreme, in example A, students are required to experiment by changing the size of the circle radius and measuring the resulting circumference. This task does not require students to have knowledge of the underlying software, but only to follow simple instructions specific to the pre-designed template for the task. At the other extreme, in examples D1 and D2, carrying through the tasks requires substantial knowledge of software tools. Thus the instrumental requirements differed significantly between the tasks, calling for very different levels of preparation of students to work on these problems, and correspondingly of teacher guidance.

Table 3
Software types and instrumented techniques

<i>Example</i>	<i>Software type</i>	<i>Key instrumented technique(s)</i>
A	SA	Dragging points on prepared figure [Other techniques using pencil-and-paper tools]
B1	TE	Range of construction, coloring, reorientation and comparison techniques
B2	TE	Range of construction, manipulation and scaling techniques
B3	TE	Range of construction, rotation and measurement techniques
C1	GS	Use of circle construction command Use of sector division command Use of manipulative techniques to reconfigure sectors Use of measurement commands for specific lengths and areas
C2	GS	Use of shape-specific construction commands, and of regular-polygon command to construct square Use of measurement commands for lengths and angles Use of labeling and naming techniques Use of enlarging and reducing techniques
D1	GS	Use of line and segment tools Use of parallel line tool Use of labeling and naming techniques Use of locus/trace tool Use of “set” tool for finding equation of locus line Use of algebraic functions and changing parameters of equations Use of axes and function plotting
D2	GS	Use of line, triangle tools Use of perpendicular line tool Use of “member” tool to identify points on objects Use of complex CAS programming

Analyzing the Didactical Thinking behind Dynamic Mathematical Approaches

The covering statements accompanying each of the nominated examples of successful dynamic mathematical approaches provided a basis for analyzing the didactical thinking behind such approaches.

Dynamic concepts and success features. In response to the question of what makes an approach a dynamic one, many of the responses alluded rather indirectly to the dynamic aspect, focusing on the mathematical processes involved and/or the mathematical outcomes anticipated. Related ideas often appeared in response to the question on what features of the approach lead to it being particularly successful, and how they contribute to this success. Because participants’ replies to these two questions were similar in many respects, we have treated them together. Table 4 shows those ideas identified in responses from at least two of the contributing organizations.

Focusing directly on the dynamic aspect and its more immediate affordances, the core idea appears to be one of manipulating displayed representation(s) to highlight associated variation and non-variation of properties, and relations between different

states/representations. Employing this type of user interaction to support visualization and observation is seen as creating an environment that encourages exploration and experimentation through which mathematical properties and relationships can be discovered. More broadly, use of computers is seen as improving efficiency and precision of construction, measurement and computation, and as enhancing participation by students and interaction between them. Indeed, respondents considered that it was the use of computers that made many of the nominated examples possible – or feasible – in ordinary teaching circumstances.

Table 4
Dynamic concepts and success features

<i>Coded idea</i>	<i>Examples</i>
Variation and non-variation	[D]ynamic constructions can be varied on the screen. [A] In the activity for the properties of quadrilaterals the participants can actually see the properties of a shape since these do not change when one drags the shapes. [C2] Image is not static, students can create something themselves, and there is a change in it. [D1]
Visualization and observation	The dynamic construction helps to see the circumference as a one-dimensional line with a length that can be compared with the diameter. [A] The virtual environment <i>Origami Nets</i> ensures dynamic visualization. The 2D-3D movement can be observed and vice versa. It is possible to transform one constructed figure into another. [B2] In activity with the area of a circle, students can actually see where the formula for the area of a circle arises from. When students cut the circle in a number of sectors and place them in a certain way, they can see that the figure starts resembling to a quadrilateral. This becomes more obvious as the number of sectors in which a circle is cut into becomes larger. [C1] We want students, when we offer them problems, to use the possibility of visualization [...] Working individually, they can discover some of the characteristics of quadratic functions at the same time. During the computer experiment with the quadratic function, they can – besides other things – observe the role of parameters and the sense of the constant term in the standard form of the quadratic function notation. [D1]
Exploration and discovery	With the dynamic worksheet, students can explore and discover the relationship between the circumference and the diameter of circles. [A] This task is interactive since students are active learners; they construct and measure the dimensions of a parallelogram and extract their conclusions. This interaction with the shape and measurements allows students to draw their own conclusions. [C1] We are confident, that by experimentation, observation of the situations, which they know from real life, by searching hidden regularity, by the gradual creation of correct terminology, by the pacific leading to correct locution and called the things, lead the students to get an opinion that Math is a powerful tool which helps us with knowledge of the real world. [D1]

Table 4 (cont.)

<i>Coded idea</i>	<i>Examples</i>
Efficiency and precision	The measured values are more precise than measurements of objects from real life. [A] Constructions done in sufficiently short time. [B1] It also allows them to simultaneously read the measurements and see the shape. [C1] The software also allows students to do this procedure much faster and more accurately. [C2] Precise constructions/measurement. [D2]
Feasibility	With the dynamic construction one can get measured values in regular steps. [...] With real life objects one can hardly get such equidistant steps of the diameter. [A] The instruction uses material, virtual and ideal (imaginary) solids. [B2] Creates tasks that cannot be posed without the software. They were asked in the quadrilateral activity to construct a shape that carried certain properties. The dragging facility of the software allows the participants to check whether certain shapes possess specific properties. [C2]
Participation and interaction	It is possible to make competitions - to model a certain solid in terms of shape and/or volume (surface area); to do project work. [B3] Gives experimental environment where all students (including weak students) participate. All students were able to construct, measure and explore various quadrilaterals and their properties. Teachers commented that weak students were also able to experiment, discuss and participate in this activity more than they usually do with paper and pencil tasks. [C1/C2]

Teacher Actions and Potential Obstacles to Success

Responses were less developed and more varied in focus on the final questions regarding what preparation or actions or guidance by the teacher are important to the success of the approach, and what potential obstacles there are to the approach being successful and how these could be avoided or overcome. In Table 5, we offer an overview of the ideas expressed in the responses. In contrast with the detailed treatment of mathematical and instrumental issues in the specifications for the examples, responses to these questions were framed in much more general terms rarely linked to the specificities of the task. There was little evidence of knowledge gleaned directly from classroom use of the nominated examples by participating teachers, or of task-specific formulation of the types of implementation issues that teachers might encounter.

With the exception of organization B, respondents considered that using dynamic approaches required considerable preparation on the part of teachers to find or create suitable teaching ideas, and to develop suitable lesson plans and supporting resources. The preparation and use of dynamic resources called not only for mathematical and pedagogical expertise on the part of the teacher, but for the technical knowledge that enabled mathematical and pedagogical ideas to be operationalized through use of dynamic mathematical tools. Equally, decisions had to be made about how to orchestrate classroom activity appropriately and to prepare students to meet the technical demands of software use.

We note that these latter issues seemed to be largely left to the teacher to address, as the supporting materials for the nominated examples focused primarily on mathematical tasks and instrumented techniques.

Table 5

Teacher actions and potential obstacles to success

<i>Coded idea</i>	<i>Examples</i>
Preparation of lessons and resources	Producing or finding material; formulating exercises; designing learning environments [A] Preparation of worksheets. [B1/B2] Teachers need to know how to construct an activity with the use of technology and what type of questions to ask. [C1/C2] Teachers must be prepared to construct different types of lesson plans and activities to the ones that they used to prepare when students were using only paper and pencil. [C1/C2] Good examples. [D2]
Functioning of technical facilities	Coping with technical obstacles [A] The teacher must have installed the programmes on the PCs. [B1/B2/B3] The main obstacles that the teachers may face are technical problems that could arise from the use of software or computers. [C1/C2]
Technical skills of students	Giving technical support to students. [A] Students cannot handle the software properly [D2]
Orchestration of classroom activity	Arranging students' individual and cooperative work. Arranging students' presentations and discussions. [A] The difficult thing during the first two classes is to organize such individual and group work as to ensure optimal opportunities for most students. [B3] In these activities teacher needs to act as a facilitator, while students must be in control of the exploration. [C1/C2]
Teacher mathematical knowledge	It is important for teachers to know the mathematical context. [C1/C2] Good understanding of mathematics (geometric) substance of the problem. [D1]
Teacher technical knowledge (for mathematical and pedagogical purposes)	In meetings at the University or at schools teachers acquire technical skills for using DGS and for creating dynamic worksheets. [A] Teacher to know why to use this software to teach the specific mathematical concept and how to use the software. [C1/C2]
Teacher pedagogical knowledge	Be familiar and proficient in mathematics programs in the context of their mathematical knowledge [D1] [Teachers] get acquainted with general ideas and theories of teaching and learning as well as with techniques for constructing learning environments for individual and cooperative learning. [A] The main emphasis should be placed on the adequacy and effectiveness of educational practices, rather than the gratuitous use of technology. [D1]

Conclusion

This analysis of the examples of the use of dynamic mathematics that were nominated as successful by teacher educators helps to define a 'conceptual space' of crucial issues that

deserve discussion amongst teacher educators in relation to the design of programs of professional development, and often also between teacher educators and teachers in the course of such professional development.

An issue of specific relevance to teacher educators is the desirable balance and appropriate management of examples at differing mathematical levels. We have seen that the majority of the examples chosen appear to be at a mathematical level intended for use with school students. However, effective professional development needs to consider how teachers' responses to such tasks may differ from those of their students. Unlike students, teachers are generally already familiar with the (type of) result they are intended to 'discover' or the (type of) problem they are asked to 'solve'; this already established framework shapes their interpretation of the situation. Thus, in line with Laborde's (2001) findings, it may be particularly important for teachers to have some form of direct experience of how students tackle such tasks in the ordinary classroom, and to be given opportunities to reflect on such experience and build models of crucial didactical processes and variables. A much more rare approach was to engage teachers in mathematical tasks at a higher mathematical level (closer to the limits of their own mathematical knowledge). Here, it seems particularly important that teacher educators consider how to bridge between such experience and teachers' awareness of pedagogical processes and development of didactical models that can be used productively with school students.

A second issue, important for teacher educators to help teachers consider, is the variety and balance of task types. We have seen that by far the most common type of mathematical goal within the nominated examples is one of quantitatively formulating the mathematical relationship expressed by a visual representation; Monaghan (2004) described similar sets of goals. A less common goal involved the coordination of mathematical representations. Indeed, we have seen that the predominant reasoning structure within the nominated examples reflected a guided-discovery orientation grounded in directed action (to create what can be termed a didactic situation), in contrast to a problem-solving orientation grounded in constrained solution (to create an adidactic situation). While respondents argued that these tasks became more feasible, occasionally indeed possible, through use of technology, virtually all belong to the earlier levels of Laborde's (2001) typology, in the sense that the mathematical question posed does not depend on dynamic mediation. This, then, appears to be an area that would benefit from the development of wider-ranging examples and example-types.

A third issue is the handling of instrumental demands. We have seen how most of the nominated examples called for quite extensive instrumented technique on the part of students, often supported by just-in-time prompts on worksheets. The levels of instrumental demand were somewhat less where a tailored environment was available to support work around a particular topical theme, and somewhat greater where students used generic software from scratch. Even where students were provided with a prepared dynamic figure to manipulate, a degree of instrumented technique was still required for students to make mathematical sense of the figure under manipulation. Equally, only one example incorporated parallel tasks using tool systems based on pencil-and-paper and manipulatives: the dominant model for task design was one in which established tools are replaced, rather than complemented, by dynamic software. This suggests that it would be beneficial for issues of instrumental genesis

and orchestration (Trouche & Drijvers, 2008), in particular those of double instrumentation (Ruthven, 2009), to receive explicit attention within professional development.

A final issue is that the didactical thinking behind these examples appeared to be primarily *a priori*, with a focus on mathematical analysis and associated didactical reasoning. There was little evidence of direct account being taken *a posteriori* of feedback from participating teachers on classroom use of the nominated examples. While the respondents did identify, in general terms, a range of necessary actions for, and potential obstacles to, successful classroom use, there was less evidence of these being followed through in detail in the design and documentation of the nominated examples. This links back to the first issue. If the experience that teachers have during professional development is overly idealized, and insufficiently related to crucial matters that arise in classroom implementation, this represents a fundamental flaw in the process of intermediation that professional development aims to accomplish.

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Appendix

Box 1: Illustrative Task Sequence for Example A

Dynamic Worksheet - Mozilla Firefox

Dynamic Worksheet

Circumference

GEONET

diameter $d = 2.0 \text{ cm}$

total circumference $u = 6.28 \text{ cm}$

unwound circumference $s = 1.91 \text{ cm}$

Exercise 1

Here you see a circle. Move the points and note down your observations in your exercise book.

Exercise 2

Vary the diameter d of the circle and observe the circumference u . Note down the values of d and u in a table in your exercise book. Can you discover something?

Exercise 3

Measure the diameter and the circumference of several circular objects (e.g. bucket, dish, glass, tin, CD, coin, ...). Note down the measured values in your table of exercise 2.

Exercise 4

Depict your measured values in a coordinate system in your exercise book. What do you find about the relation between the circumference and the diameter of circles? Note down your considerations in your exercise book.

Exercise 5

Calculate the quotient $u:d$ of the pairs of measured values. Note down the results in your table of exercise 2. Describe your observations in your exercise book.

Exercise 6

Find a formula for the relation between the circumference and the diameter of a circle. Note down all your considerations in your exercise book.

Box 2: Illustrative Task Sequence for Example B1

Explorations with configurations of colored unit cubes

The Challenge:

Construct a $6 \times 6 \times 6$ cube by using 36 red, 36 blue, 36 green, 36 black, 36 white and 36 yellow unit cubes so that each column of the cube of size $1 \times 1 \times 6$ contains one unit cube of each color.

Task 2. Construct a $2 \times 2 \times 2$ cube using 4 blue and 4 yellow unit cubes so that each $1 \times 1 \times 2$ cuboid contains 1 blue and 1 yellow unit cube.

Task 3. Construct a $3 \times 3 \times 3$ cube by using 9 red, 9 blue and 9 yellow unit cubes so that each column of size $1 \times 1 \times 3$ contains one unit cube of each color.

Hint 1. This problem is similar to the problem of filling the unit squares in a 3×3 square with numbers from 1 to 3 so that each number appears just once in any row or any column. Use the analogy!

Hint 2. Figure out how many cuboids of size $1 \times 1 \times 3$ are there in a cube of size $3 \times 3 \times 3$.

Hint 3. Add a layer to the construction below so as to get a cube $3 \times 3 \times 3$ that satisfies the requirements.

Task 4. Construct a $4 \times 4 \times 4$ cube by using 16 red, 16 blue 16 green and 16 yellow unit cubes so that each column of size $1 \times 1 \times 4$ contains one unit cube of each color.

...

Task 31. Change the configurations given below so that each layer (in any direction) contains a cube of each participating color by exchanging two unit cubes.

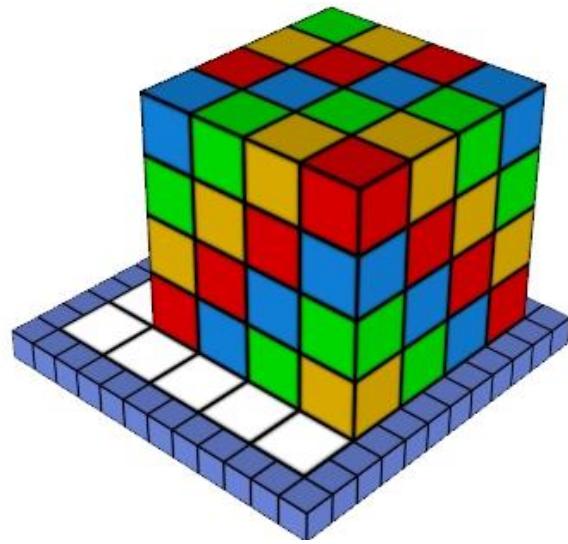


Figure 1

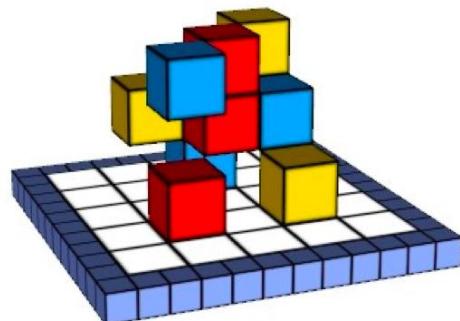


Figure 2

Box 3: Illustrative Task Sequence for Example B2

Explorations with the Origami software

The Challenge:

Construct models of the figures with the Origami program.

Task 1

To construct a figure of squares: select a square from the figure menu with the mouse pointer, select a side of the square to which to put the next square, continue until you get the desired figure.

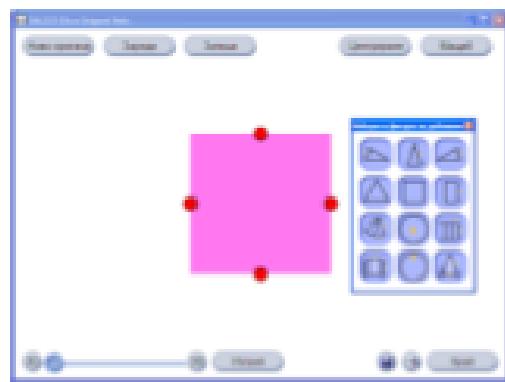


Figure 1

Task 2

To fold up the figure: select the 3D button point the square you want to bend, enter the folding angle in degrees (for example, 90)

To view the folding effect, use the slider between the 2D and 3D buttons.

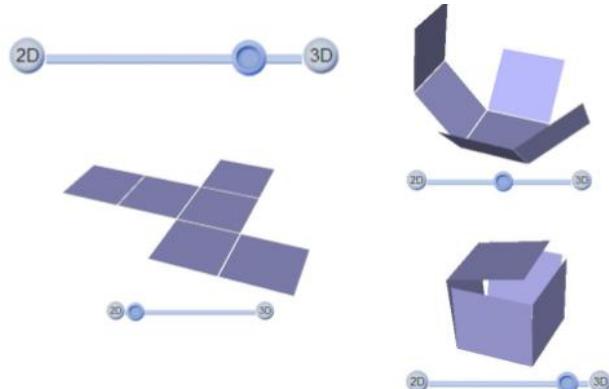
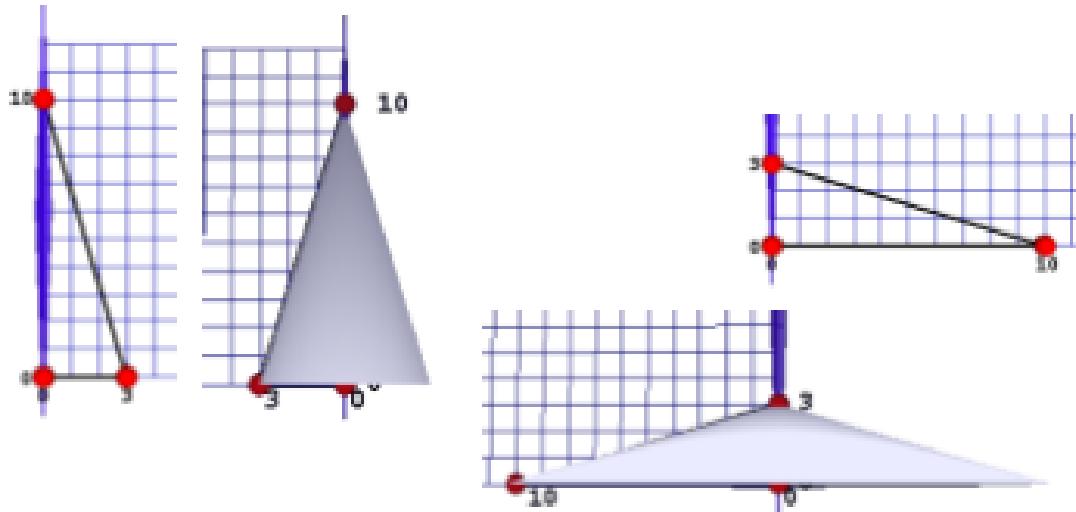


Figure 2

Box 4: Illustrative Task Sequence for Example B3

Explorations with the Math Wheels software

Construct a model to help you explain your solution, using *the Math Wheel*.



In the table below you have dimensions of a cone in each column (r – radius and h – height). Using the *Math Wheel* software fill in the first four columns.

Make your conjecture for the numbers in the last two columns. It will be easier if you convert the decimal fraction into common fractions.

Using the known formulas examine the validity of your conjecture.

Make your conclusion about how the cone volume changes when its height changes.

Cone

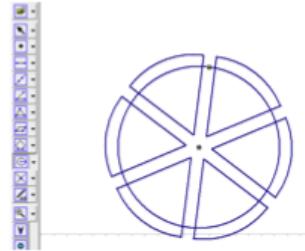
r	10	10	10	10	10	10
h	1	2	4	8	16	20
V						

Box 5: Illustrative Task Sequence for Example C1

Explorations with the EucliDraw software

The Challenge:

In this activity, students will use the dynamic software Euclidraw Jr to discover the formula for the area of the circle. The method that we used to discover the formula of the circle is similar to the methods used by Eudoxos and Archimedes.



Task 1

Construct a circle on the screen. Then, using the Divide in Sectors tool, divide the circle into six sectors. Put the sectors, in such a way as to construct a parallelogram.

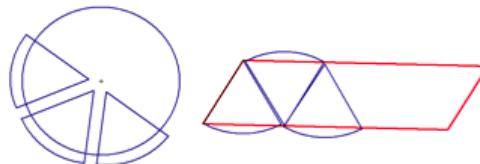


Figure 1

Task 2

Repeat the first task, by dividing the circle into eight, ten, twelve, and twenty sectors. Every time measure its circumference and the length of its radius. Complete the table below.

	Area of circle	Circumference	Radius	Base of parallelogram	Height of parallelogram	Area of parallelogram	$\frac{\text{radius} \times \text{circumference}}{2}$
Six sectors							
Eight sectors							
Ten sectors							
Twelve sectors							
Twenty sectors							

Figure 2

Box 6: Illustrative Task Sequence for Example C2

Properties of quadrilaterals

In this activity we used the dynamic software of EucliDraw Jr to investigate different quadrilaterals and their properties.

Section A

1. Select and open the tool  . Then construct a rectangle, a square, a parallelogram, a trapezium and a rhombus.

(You can construct the square using the *Regular Polygons* tool .

2. Write below each shape, its name by using the Text tool which is on the left side of the screen.

Complete the table below, using the Measurement tools.

	Opposite sides are equal	Opposite angles are equal	All sides are equal	All angles are equal
Rectangle				
Square				
Parallelogram				
Trapezium				
Rhombus				

3. Write your observations below.
4. Check your observations by enlarging and reducing your shapes with the use of your mouse. Are the observations that you wrote above still valid for all cases?

Section B

1. Construct the quadrilateral which has all the properties listed below:
 - All sides have equal length.
 - Its angles are not right angles.
 - Its diagonals are perpendicular.
 - Opposite sides are parallel.
2. Construct an isosceles trapezium and investigate its properties.

Write the properties of an isosceles trapezium.

Box 7: Illustrative Task Sequence for Example D1

Creating a concept of quadrilateral function

What mutual characteristics do line segments ZP , $X P'$, and YP'' have? (See Figure 1)

Look at the picture on the computer. What is the curve on which point V moves if we move point Z along a line parallel to line XY ? What is point V ? Try to estimate what the shape of the curve is.

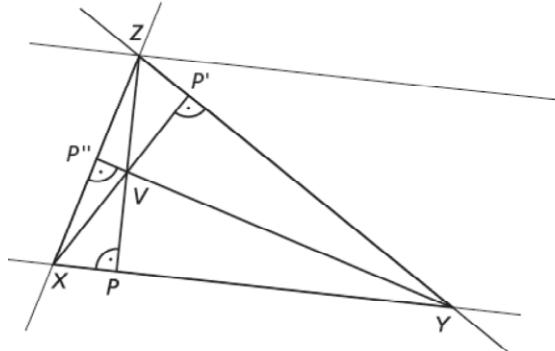


Figure 1

1. Using any DGS, draw a horizontal line p and mark a line segment AB on the line (See Figure 2). Draw a line q parallel to the line p . Choose a point C on the line q . Construct a triangle ABC and draw intersection V of its heights. Move a point C along the line q and observe the curve formed by a point V . Use tool Trail yes/no.

Using the button Set in the program Cabri, draw a curve indicating the movement of a point V when the position of a point C is changed. Display axis and grid points in the monitor pad. [...]

2. What do parameters a , b , c influence? How is the curve shape changing when the parameters are changing?

How would you name the given function when you can see the equation describing it? Do you know what the curve formed by the point V is called?

3. Use computer and draw the graphs of some quadratics functions. Parameter changes!

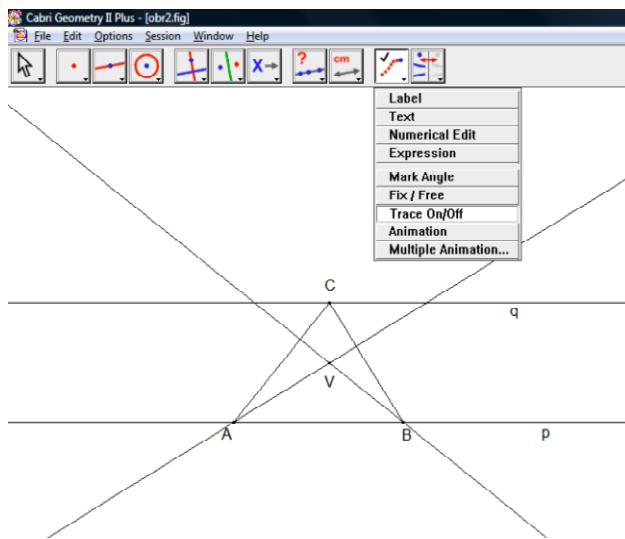


Figure 2

Box 8: Illustrative Task Sequence for Example D2

Creating a concept of quadrilateral function

The challenge:

Show that the statement “Three heights of a triangle are concurrent” is true.

Task 1

Demonstrate the situation of the statement in a geometry system, or equivalently, draw the situation using paper and pencil, Figure 1.

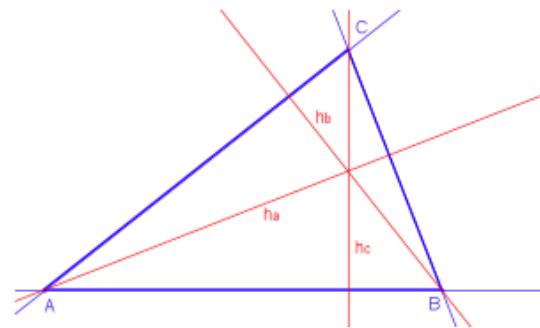


Figure 1

Task 2

Demonstrate the statement in a dynamic geometry system, Figure 2.

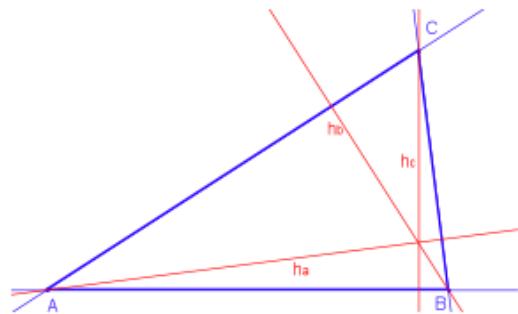


Figure 2

Task 3

Verify the statement in DGS. We could act as follows, Figure 3:

- denote the intersection of the heights by P. (Does the point P always exist? Why?).
- verify whether P also lies on the remaining height.

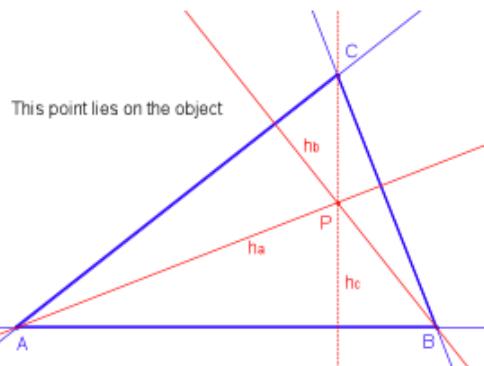


Figure 3