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# The Quaterniontonic and Octoniontonic Fibonacci Cassini's Identity: An Historical Investigation with the Maple's Help

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**ABSTRACT**

This paper discusses a proposal for exploration and verification of numerical and algebraic behavior correspondingly to Generalized Fibonacci model. Thus, it develops a special attention to the class of Fibonacci quaternions and Fibonacci octonions and with this assumption, the work indicates an investigative and epistemological route, with assistance of software CAS Maple. The advantage of its use can be seen from the algebraic calculation of some Fibonacci's identities that showed unworkable without the technological resource. Moreover, through an appreciation of some mathematical definitions and recent theorems, we can understand the current evolutionary content of mathematical formulations discussed over this writing. On the other hand, the work does not ignore some historical elements which contributed to the discovery of quaternions by the mathematician William Rowan Hamilton (1805 – 1865). Finally, with the exploration of some simple software's commands allows the verification and, above all, the comparison of the numerical datas with the theorems formally addressed in some academic articles.

**Keywords:** Fibonacci's model, historical investigation, Fibonacci quaternions, Fibonacci octonions, CAS Maple

## INTRODUCTION

Undoubtedly, the role of the Fibonacci's sequence is usually discussed by most of mathematics history books. Despite its presentation in a form of mathematical problem, concerning the birth of rabbits' pairs, still occurs a powerful mathematical model that became the object of research, especially with the French mathematician François Édouard Anatole Lucas (1842–1891). From his work, a profusion of mathematical properties became known in the pure mathematical research, specially, from the sixties and the seventies.

With the emergence of the periodical *The Fibonacci Quarterly*, we register the force of the Fibonacci's model, with respect to their various ways of generalization and specialization. Thus, we can indicate the works of Brother (1965), Brousseau (1971), Horadam (1963; 1967). From these works, besides the well-known the second order recurrence formula $f_{n+2} = f_{n+1} + f_n, n \geq 0$, we also derive the following identity $f_n = (-1)^{n+1} f_n = (-1)^{n-1} f_n$, for any integer 'n'.

Moreover, other studies found other ways for the process of generalization of the Fibonacci's model. Some of them employ methods of Linear Algebra (King, 1968; Waddill & Sacks, 1967), while others explore the theory of polynomial functions (Bicknell-Johnson & Spears, 1996; Tauber, 1968). In some articles, experts are interested in the extent of the fibonacci function in other numerical fields, like real numbers and complex numbers, relatively to its set of subscripts (Reiter, 1993; Scott, 1968).

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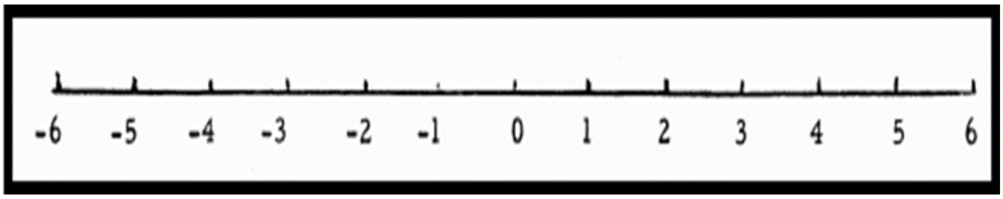


**Figure 1.** Brousseau (1965) discusses the extension's process to the Fibonacci's model

From some trends work around the model, we express our interest in involving the process of Fibonacci's complexification model and the corresponding introduction of imaginary units. From the historical point of view, we can recall the Italian mathematician's work Corrado Segre (1863 – 1924), involving the mathematical definition of the bicomplex numbers, indicated by $\{z_1 + z_2 \mathrm{j} | z_1 = \mathrm{a} + \mathrm{bi}, z_2 = c + di, \mathrm{j}^2 = -1\}$ . But, every conceptual element in the previous set may be still expressed as $z_1 + z_2 \cdot \mathrm{j} = \mathrm{a} + \mathrm{b} \cdot \mathrm{i} + \mathrm{c} \cdot \mathrm{j} + \mathrm{d} \cdot \mathrm{i} \cdot \mathrm{j}$, with the real numbers $a, b, c, d \in IR$. This abstract entity may also be represented by $\mathrm{a} + \mathrm{b} \cdot \mathrm{i} + \mathrm{c} \cdot \mathrm{j} + \mathrm{d} \cdot k$, with the set of operational rules $i^2 = j^2 = -1, ij = ji = k$.

Given these elements and others that we will seek to discuss in the next sections, mainly some elements with respect to an evolutionary epistemological trajectory and, especially, an historical perspective. In this way, it may raise an understanding about the continued progress in Mathematics and some elements, which can contribute to an investigation about the quaternions and octonions of Fibonacci's sequence which is customarily discussed in the academic environment, however in relation to their formal mathematical value.

Moreover, in view of the use of software Maple, we will explore particular situations enabling a heuristic thought and not completely accurate and precise with respect to certain mathematical results. Such situations involve checking of algebraic properties extracted from current numerical and combinatorial formulations of the Fibonacci's model.

Thus, in the next section, we consider some elements and properties of quaternions or hyper-complex numbers (Kantor & Solodovnikov, 1989).

## SOME HISTORICAL ASPECTS ABOUT THE QUATERNIONS

In general, a quaternion is a hyper-complex number and is defined by the following equation $q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_4$, where the coefficients $q_0, q_1, q_2, q_3$ are real, and the set $\{e_0, e_1, e_2, e_4\}$ is the canonical base for the $IR^4$. Here, it satisfies the following rules ${e_2}^2 = {e_3}^2 = {e_4}^2 = -1, e_2 e_3 = e_4 = -e_3 e_2$. The conjugate of the quaternion $q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_4$, is defined by $q^* = q_0 e_0 - q_1 e_1 - q_2 e_2 - q_3 e_4$ and the norm $N(q) = {q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2$.

Halici (2015, p. 1) comments that the quaternions are a number system which extends to the complex numbers, first introduced by Sir William Rowan Hamilton (1805 – 1865), in 1843. From the fact $e_2 e_3 = e_4 = -e_3 e_2$ we say that the algebra $IH$ is not commutative but associative, as, $(e_2 e_3)\mathrm{e}_4 = e_2(e_3 e_4)$. On the other hand, the historical process, concerning the systematization process of quaternions demanded considerable time and effort, above all, the capacity of Hamilton 's imagination. We can see this from the comments due to Hanson (2006, p. 5).

> *Quaternions arose historically from Sir William Rowan Hamilton's attempts in the midnineteenth century to generalize complex numbers in some way that would be applicable to three-dimensional (3D) space. Because complex numbers (which we will discuss in detail later) have two parts, one part that is an ordinary real number and one part that is "imaginary," Hamilton first conjectured that he needed one additional "imaginary" component. He struggled for years attempting to make sense of an unsuccessful algebraic system containing one real and two "imaginary" parts. In 1843, at the age of 38, Hamilton had a brilliant stroke of imagination, and invented in a single instant the idea of a three-part "imaginary" system that became the quaternion algebra. According to Hamilton, he was walking with his wife in Dublin on his way to a meeting of the Royal Irish Academy when the thought struck him*

An element or factor that cannot be disregarded in the previous section concerning the role of the mathematical genius in the sense of obtaining an idea or an insight (Hadamard, 1945) in view of the consistent formulation of the set of quaternions. Moreover, an essential aspect pointed by Hanson (2006) relates precisely to the preparation and formulation of a formal mathematical definition process. We can, for example, see in

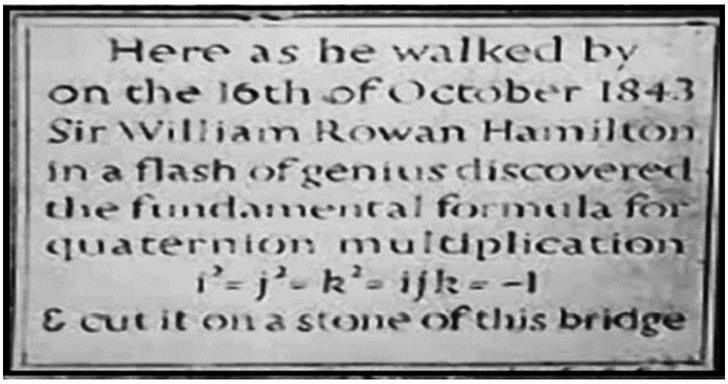


**Figure 2.** Hanson (2006, p. 6) comments the iluminatory moment that Hamilton stablishes the rules for the quaternions set

**Figure 2**, Hanson comments some explication present in a bronze plaque that mentions explanatory words about the glorious and unexpected moment, that the professional mathematician, with a view to establishing a new mathematical set which is still studied nowadays (Hanson, 2006).

## SOME HISTORICAL ASPECTS ABOUT THE OCTONIONS

In view of the formal properties of regular quaternions, mainly its dimensional properties, of course, after a certain time, a natural thought refers to increasing the dimensional set. Thus, from this dimensional elevation, occurred soon after the emergence of all octonions.

In this way, let $\Theta$ be the octonion algebra over the real number field $IR$. Keçiolioglu & Akkus (2014, p. 2) record that, from the Cayley-Dickson process, we can take any element $p \in \Theta$, therefore, it can be written as $p = p' + p''e$, where $p', p'' \in IH$ the real quaternion division algebra. On the other hand, the excerpt below shows the important role of Hamilton's student.

> *Less well known is the discovery of the octonions by Hamilton's friend from college, John T. Graves. It was Graves' interest in algebra that got Hamilton thinking about complex numbers and triplets in the first place. The very day after his fateful walk, Hamilton sent an 8-page letter describing the quaternions to Graves. Graves replied on October 26th, complimenting Hamilton on the boldness of the idea, but adding, "There is still some thing in the system which gravels me. I have not yet any clear views as to the extent to which we are at liberty arbitrarily to create imaginaries, and to endow them with supernatural properties." And he asked: "If with your alchemy you can make three pounds of gold, why should you stop there?" (Conway & Smith, 2004, p. 9)*

Graves contributed to a first impulse to obtain some preliminary ideas about octônios, however, the fundamental elements for its definitive establishment, as explained Halici (2015), was awarded by Cayley, who defined his algebra, subject to certain formal mathematical rules. In this way, Halici (2015, p. 5) records the system of multiplication: $\begin{cases} e_j^2 = -e_0, j = 0,1,2,\dots 7 \\ e_j e_k = -e_k e_j, j \neq k, j, k = 1,2,\dots,7 \end{cases}$.

Halici (2015, p. 6) comments that is well known the octonions algebra $\Theta$ is the real quaternion division algebra. Moreover, among all the real division algebras octonion algebra forms the largest normed division algebra.

In a simplified way, the notational point of view, we can write this set $\Theta = \{p = p' + p''\mathrm{e} | \mathrm{p}'\mathrm{p},'' \in \mathrm{IH}\}$. Moreover, the addition and multiplication of any two octonions, $p = p' + p''\mathrm{e}$, $q = q' + q''\mathrm{e}$ are defined by $p + q = (p' + p''\mathrm{e}) + (q' + q''\mathrm{e}) = (\mathrm{p}' + \mathrm{q}') + (\mathrm{p}'' + \mathrm{q}'')\mathrm{e}$; $p \cdot q = (\mathrm{p}'\mathrm{q}' - \overline{\mathrm{q}''}\mathrm{p}'') + (\mathrm{q}''\mathrm{p}' + p''\overline{q'})\mathrm{e}$.

From this point, with the appreciation of some definitions related Fibonacci model, we note a natural style of composition properties of the two different mathematical models. One factor that cannot be disregarded with respect to the historical time corresponding to the mathematical evolutionary process, especially to the sixties, and that contributed to the current research.

Before concluding, we recall the first articles that explored some fundamental properties, in order to formulate and define new conceptual entities. Thus, we find the following definitions.

$$
\begin{aligned}
F_{k,0} &= 0\\
F_{k,1} &= 1\\
F_{k,2} &= k\\
F_{k,3} &= k^2 + 1\\
F_{k,4} &= k^3 + 2k\\
F_{k,5} &= k^4 + 3k^2 + 1\\
F_{k,6} &= k^5 + 4k^3 + 3k\\
F_{k,7} &= k^6 + 5k^4 + 6k^2 + 1\\
F_{k,8} &= k^7 + 6k^5 + 10k^3 + 4k\\
&\dots
\end{aligned}
$$

**Figure 3.** Falcon (2014, p. 149) discusses some properties related to the k-Fibonacci numbers

**Definition 1**: The $n$th Fibonacci quaternions is defined by $Q_n = f_n e_0 + f_{n+1} e_1 + f_{n+2} e_2 + f_{n+3} e_4$, where the coefficients are de Fibonacci numbers. (Horadam, 1963).

Sometime later, we find a definition that involves the complexity of the process of the Fibonacci numbers and, respectively, a concern with their representation in the complex plan. We observe such a characterization in the next definition.

**Definition 2**: The complex Fibonacci numbers are by $C_n = f_n + f_{n+1} i$, where the coefficients are de Fibonacci numbers and the imaginary unit $i^2 = -1$. (Jordan, 1965).

From the definition 1, we can determine the following particular values of all quaternions: $Q_1 = e_0 + e_1 + 2e_2 + 3e_4$, $Q_2 = e_0 + 2e_1 + 3e_2 + 5e_4$, $Q_3 = 2e_0 + 3e_1 + 5e_2 + 8e_4$, $Q_4 = 3e_0 + 5e_1 + 8e_2 + 13e_4$, $Q_5 = 5e_0 + 8e_1 + 13e_2 + 21e_4$, $Q_6 = 8e_0 + 13e_1 + 21e_2 + 34e_4$, $Q_7 = 13e_0 + 21e_1 + 34e_2 + 55e_4$, $Q_8 = 21e_0 + 34e_1 + 55e_2 + 89e_4$, $Q_9 = 34e_0 + 55e_1 + 89e_2 + 144e_4$, $Q_{10} = 55e_0 + 89e_1 + 144e_2 + 233e_4$, $Q_{11} = 89e_0 + 144e_1 + 233e_2 + 377e_4$, etc.

Moreover, some preliminary values for the complex Fibonacci numbers are $C_0 = 0 + i, C_1 = 1 + i, C_2 = 1 + 2i, C_3 = 2 + 3i, C_4 = 3 + 5i, C_5 = 5 + 8i, C_6 = 8 + 13i$, etc. Most recently, we find the k-Fibonacci's definion numbers.

**Definition 3**: The sequence of the k-Fibonacci numbers are defined by the recurrence relation $F_{k,n+1} = kF_{k,n} + F_{k,n-1}, F_{k,0} = 0, F_{k,1} = 1, n \geq 1, k \in IR$. (Fálcon & Plaza, 2007). (see **Figure 3**).

**Definition 4**: The k-Fibonacci quaternion is defined by the recurrence relation $Q_{k,n} = F_{k,n} + F_{k,n+1} i + F_{k,n+1} j + F_{k,n+2} k$, with $n \geq 0$. (Ramirez, 2015, p. 204).

The collection of these mathematical definitions should convey to the reader an understanding about the evolutionary process of the Fibonacci model, shown adhered to the other models discussed, even like the quaternions and octonions. Soon after, we discuss in detail some invariants mathematical properties, when we examine closely each of the previous mathematical definitions.

## THE FIBONACCI QUATERNIONS AND FIBONACCI OCTONIONS' RESEARCH

Many investigations have developed a profusion of properties relatively the mathematical definitions commented in the last section. In addition, other mathematicians are still interested in the dual form of quaternions (Nurkan & Güven, 2015) and octonions (Savin, 2015) or the split of quaternions or octonions that we will not discuss here (Halici, 2015). Similarly to what happened way in the sixties, with properties related to extension of the subscripts to the integers numbers, Halice (2012), explains the determination of the set of numbers $\{Q_{-n}\}_{n \in IN}$.

From a corollary discussed by Halice (2012), we can determine the quaternions numbers, with negative subscripts, through the following substitution $f_n = (-1)^{n+1} f_n$, we can easily get $Q_{-n} = f_{-n} e_0 + f_{-(n-1)} e_1 + f_{-(n-2)} e_2 + f_{-(n-3)} e_4$, follow that $Q_{-n} = (-1)^{n+1} f_n e_0 + (-1)^n f_{(n-1)} e_1 + (-1)^{n-1} f_{(n-2)} e_2 + (-1)^{n-2} f_{(n-3)} e_4$, or yet $Q_{-n} = (-1)^{n+1} (f_n e_0 - f_{n-1} e_1 + f_{n-2} e_2 - f_{n-3} e_4)$. Finally, from this formula, we determine: $Q_0 = 0e_0 + e_1 + e_2 + 2e_4$, $Q_{-1} = 1e_0 + 0e_1 + 1e_2 + 1e_4$, $Q_{-2} = -1e_0 + 1e_1 + 0e_2 + 1e_4$, $Q_{-3} = 2e_0 - 1e_1 + 1e_2 + 0e_4$, $Q_{-4} = -3e_0 + 2e_1 - 1e_2 + 1e_4$, $Q_{-5} = 5e_0 - 3e_1 + 2e_2 - 1e_4$, $Q_{-5} = 5e_0 - 3e_1 + 2e_2 - 1e_4$, $Q_{-6} = -8e_0 + 5e_1 - 3e_2 + 2e_4$, $Q_{-7} = 13e_0 - 8e_1 + 5e_2 - 3e_4$, $Q_{-8} = -21e_0 + 13e_1 - 8e_2 + 5e_4$, $Q_{-9} = 34e_0 - 21e_1 + 13e_2 - 8e_4$, $Q_{-10} = -55e_0 + 34e_1 - 21e_2 + 13e_4$, $Q_{-11} = 89e_0 - 55e_1 + 34e_2 - 21e_4$, Moreover, in the studies concerned by the k-Fibonacci numbers (see definition 3), some them indicate unexpected properties (Fálcon, 2014; 2016). From the definition, we can list some of their initial values, as can be seen below.

From the algebraic expression and by using the mathematical definition formulated by Ramirez (2015), we can still obtain: $Q_{k,1} = 0 + i + kj + (k^2 + 1)k$, $Q_{k,2} = k + (k^2 + 1)i + (k^3 + 2k)j + (k^4 + 3k^2 + 1)k$, $Q_{k,3} = (k^2 + 1) + (k^3 + 2k)\mathrm{i} + (k^4 + 3k^2 + 1)\mathrm{j} + (k^5 + 4k^3 + 3k)k$, $Q_{k,4} = (k^3 + 2k) + (k^4 + 3k^2 + 1)\mathrm{i} + (k^5 + 4k^3 + 3k)\mathrm{j} + (k^6 + 5k^4 + 6k^2 + 1)k$, $Q_{k,5} = (k^4 + 3k^2 + 1) + (k^5 + 4k^3 + 3k)\mathrm{i} + (k^6 + 5k^4 + 6k^2 + 1)\mathrm{j} + (k^7 + 6k^5 + 10k^3 + 4k)k$, etc. (see definition 4).

Before concluding the current session, we will announce some theorems that constitute a derivation and generalization of Cassini's identity, formulated by italian mathematician Giovanni Domenico Cassini (1625 – 1712) (Koshy, 2007). We note, however, that almost of these identities can be proved by mathematical induction. The first one characterizes a quaterniotonic version for the classical Cassini's identity.

**Theorem 1**: Let $Q_n$ be the Fibonacci Generalized Quaternion. Then, the following formula is the Cassini's identity for this number ($n \geq 1$). (Akyigit, Kösal, & Todun, 2014, p. 640).$Q_{n-1}Q_{n+1} - {Q_n}^2 = (-1)^n((1 + \alpha - \beta + \alpha\beta) + (1 + \beta)i + (3 + \alpha)j + 3k)$, where $\alpha, \beta \in IR$.

**Corolary 1**: For particular numbers $\alpha = 1 = \beta$ we have $Q_{n-1}Q_{n+1} - {Q_n}^2 = (-1)^n(2Q_1 - 3k)$ (Ramirez, 2015).

**Theorem 2**: Let $Q_n$ be the Fibonacci Generalized Quaternion. Then, the following formula is the Cassini Identity for this number $(n \geq 1)Q_{n-1}Q_{n+1} - {Q_n}^2 = (-1)^n(2Q_1 - 3k)$. (AKYIGIT; KÖSAL & TOSUN, 2014, p. 640).

**Theorem 3**: For any integer $n$, we have $O_{n-1}O_{n+1} - {O_n}^2 = (-1)^n(T_0 - Q_0 + 14e_5 + 14e_6 + 7e_7)$. (Keçilioglu & Akkus, 2014, p. 640).

In the theorem 3, we have indicated the octiotonic Cassini's version.

**Theorem 4**: For any integer $n$, we have $Q_{n-1}Q_{n+1} - {Q_n}^2 = (-1)^n(T_0 - Q_0 + 14e_5 + 14e_6 + 7e_7)$. (Keçilioglu & Akkus, 2014, p. 640).

On the other hand, we know a strong tradition in the works in order to use the matrix approach and with the goal to derive some generalized results. Indeed, from the Halici (2012, p. 3), we consider the particular matrix $Q_{IH} = \begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix}$ called the Fibonacci quaternion matrices. From this, we naturally define the Quaternion Fibonacci matrices, described for $Q_\Theta = \begin{pmatrix} O_2 & O_1 \\ O_1 & O_0 \end{pmatrix}$, where $O_2, O_1, O_0$ are the Fibonacci octonion numbers and, similarly, we consider $Q_{k,\mathrm{IH}} = \begin{pmatrix} Q_{k,2} & Q_{k,1} \\ Q_{k,1} & Q_{k,0} \end{pmatrix}$ the k-Quaternion Fibonacci matrices, where $Q_{k,2}, Q_{k,1}, Q_{k,0}$ and the $Q_{k,\Theta} = \begin{pmatrix} O_{k,2} & O_{k,1} \\ O_{k,1} & O_{k,0} \end{pmatrix}$ the k-Octonion Fibonacci matrices, where $O_{k,2}, O_{k,1}, O_{k,0}$ are the k-Fibonacci octonions.

Surely, there are other forms of representation of Fibonacci quaternions and octonios. However, we see that its representation through a matrix, of second order, will be very useful, especially at the time of implementation of CAS Maple. By the software help, we will investigate the numerical behavior of some particular cases (Cassini's identity) and thereby conjecture a closed Cassini's formula, for some particular sets.

## HISTORICAL INVESTIGATIONS WITH THE MAPLE'S HELP

In this section, we will indicate some basic commands and a command package that let you explore a set of numerical operations and algebraic with quaternions and octonions Fibonacci. The software will allow an especially verification and numerical exploration for particular sets, with a view to determining and formalizing certain properties. The aim of this worksheet is to define some procedures in order to make computations in a Fibonacci quaternion and octonion algebras over the field of rational number. Below the figure, we see that the Maple's command package that allows us to explore a series of operations with quaternions and octonios.

The Quaternions package allows the user to construct and work with quaternions in Maple as naturally as you can work with complex numbers. The list of procedures may should be inserted

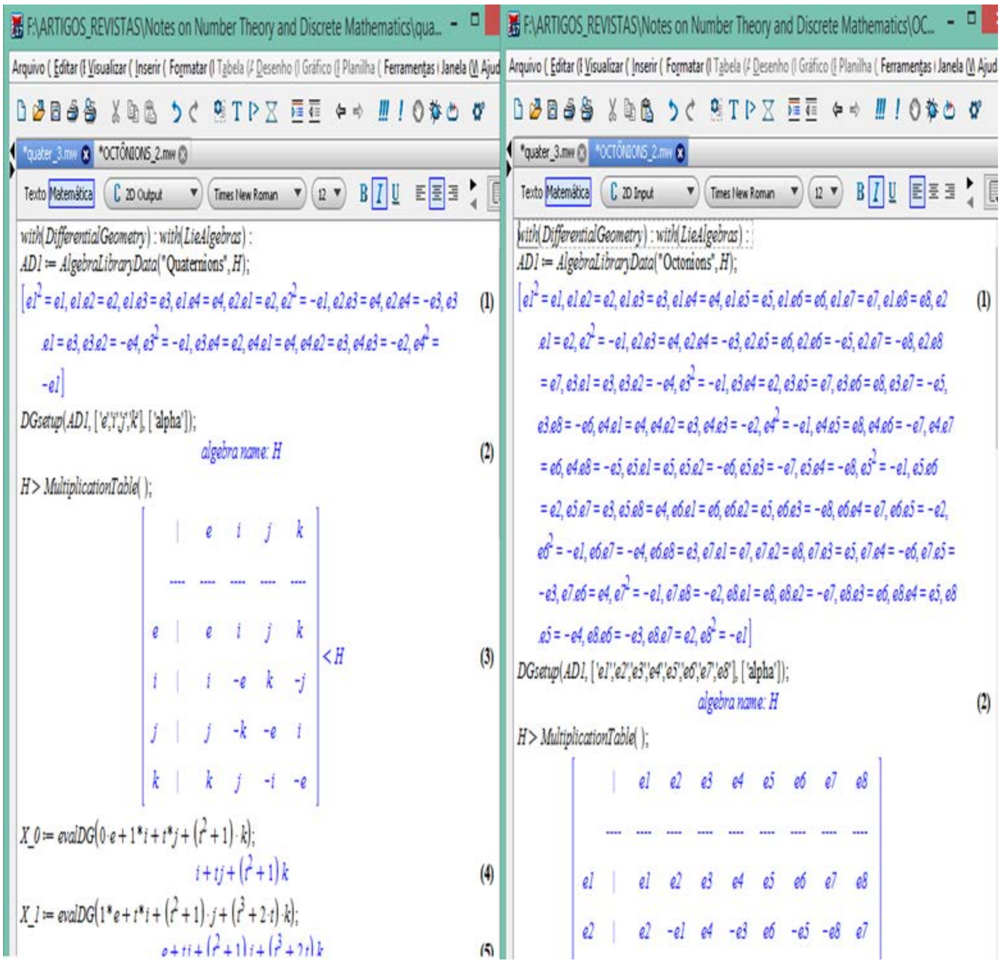


**Figure 4.** Quaternions and Octonions Maple's package

> with (DifferencialGeometry): With (LieAlgebra):

> AD1:=AlgebraLibraryData("Quaternions",H);

> DGsetup(AD!,['e','i','j','k'],['alpha']);

H>H>MultiplicationTable( );

> with (DifferencialGeometry): With (LieAlgebra):

> AD1:=AlgebraLibraryData("Octonions",H);

> DGsetup(AD!,['e1','e2','e3','e4','e5','e6','e7'],['alpha']);

H>H>MultiplicationTable( );

In the **Figure 4**, we can visualize some preliminar procedures, for the purpose of inserting some particular cases.

Following, we can declare the vectors that we want to work, taking into account a particular set. Below, we indicate the following vectors.

X_0:=evalDG(0*e+1*i+1*j+1*k); (for a Fibonacci quaternion number)

X_0:=evalDG(0*e+1*i+t*j+(t^2+1)*k); (for a k-Fibonacci quaternion number)

X_0:=evalDG(0*e1+1*e2+1*e3+2*e4+3*e5+5*e5+8*e6+13e7+21*e8); (for a Fibonacci octonion number)

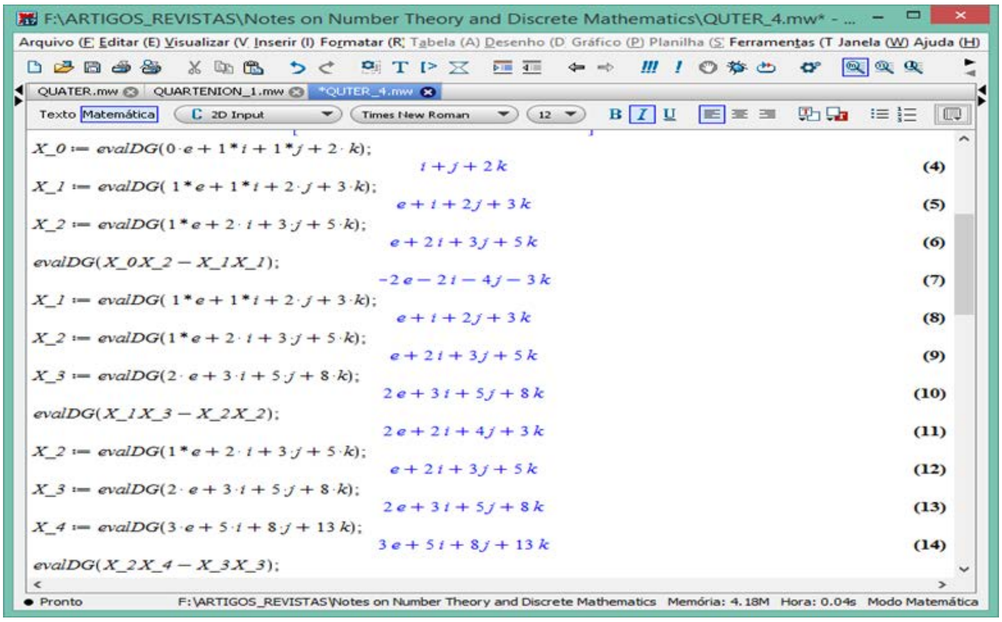


**Figure 5.** Numerical verification of Cassini's identity with the Maple

X_0:=evalDG(0*e1+1*e2+t*e3+(t^2+1)*e4+(t^3+2t)*e5+(t^4+3t^2+1)*e6+(t^5+4*t^3+3k)*e7+(t^6+5t^4+6t^2+1)*e8); (for a k-Fibonacci octonion number).

The quaternions conjugate is indicated by X_00:=DGconjugate(X_0).

This list involves all vectors we want to evaluate. Thus, in **Figure 4**, we declare the following: X_0, X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8,. etc. Soon after, we define a command to evaluate the result of the following set of operations as indicated evalDG(X_0.X_2-X_1.X_1); evalDG(X_1.X_3-X_2.X_2); evalDG(X_2.X_4-X_3.X_3); evalDG(X_3.X_5-X_4.X_4); evalDG(X_4.X_5-X_5.X_5), etc.

In all cases, we observed the invariant behavior of the result of the operation indicated. Indeed, we can distinguish the same algebraic expression $(-1)^n(2e_1+2e_2+4e_3+3e_4)$, which varies on the signal. Furthermore, we note that we can describe it as follows $(2e_1+2e_2+4e_3+3e_4)=2(e_1+e_2+2e_3+3e_4)-3e_4=2Q_1-3e_4$.

Moreover, the same verification can be appreciated, when we consider the subscripts integers. In this case, we can use the formula $Q_{-n}=f_{-n}e_0+f_{-(n-1)}e_1+f_{-(n-2)}e_2+f_{-(n-3)}e_4$ and, from this, we will indicate some specific cases and, thereby, we obtain always the same result. Indeed, in the **Figure 5**, we get the corresponding value of the expression $Q_{-n+1}Q_{-n-1}-{Q_{-n}}^2=(-1)^{-n}$. Similarly to the previous case, we have $(-1)^n(2e_1+2e_2+4e_3+3e_4)$ (**Figure 6**).

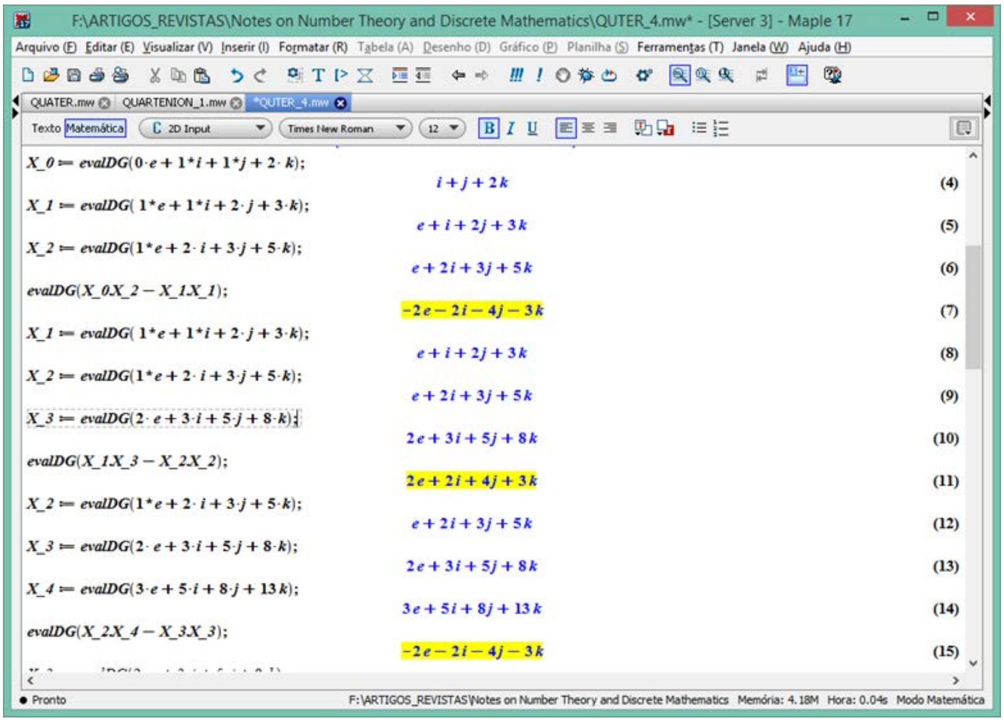


**Figure 6.** Check the corresponding values for integer indices

In the **Figure 3**, we see the values for k-Fibonacci sequence. From the **Figure 7**, we consider the corresponding set of values relatively the k-Quaternions of Fibonacci, indicated by $Q_{k,n} = F_{k,n} + F_{k,n+1}i + F_{k,n+1}j + F_{k,n+2}k$. Similarly, we take the formula $Q_{k,-n} = F_{k,-n} + F_{k,-n+1}i + F_{k,-n+1}j + F_{k,-n+2}k$. From this, we always find the expression: $2e_1 + 2e_2 + (t^3 + 4t)e_3$, where $t \in IR$.

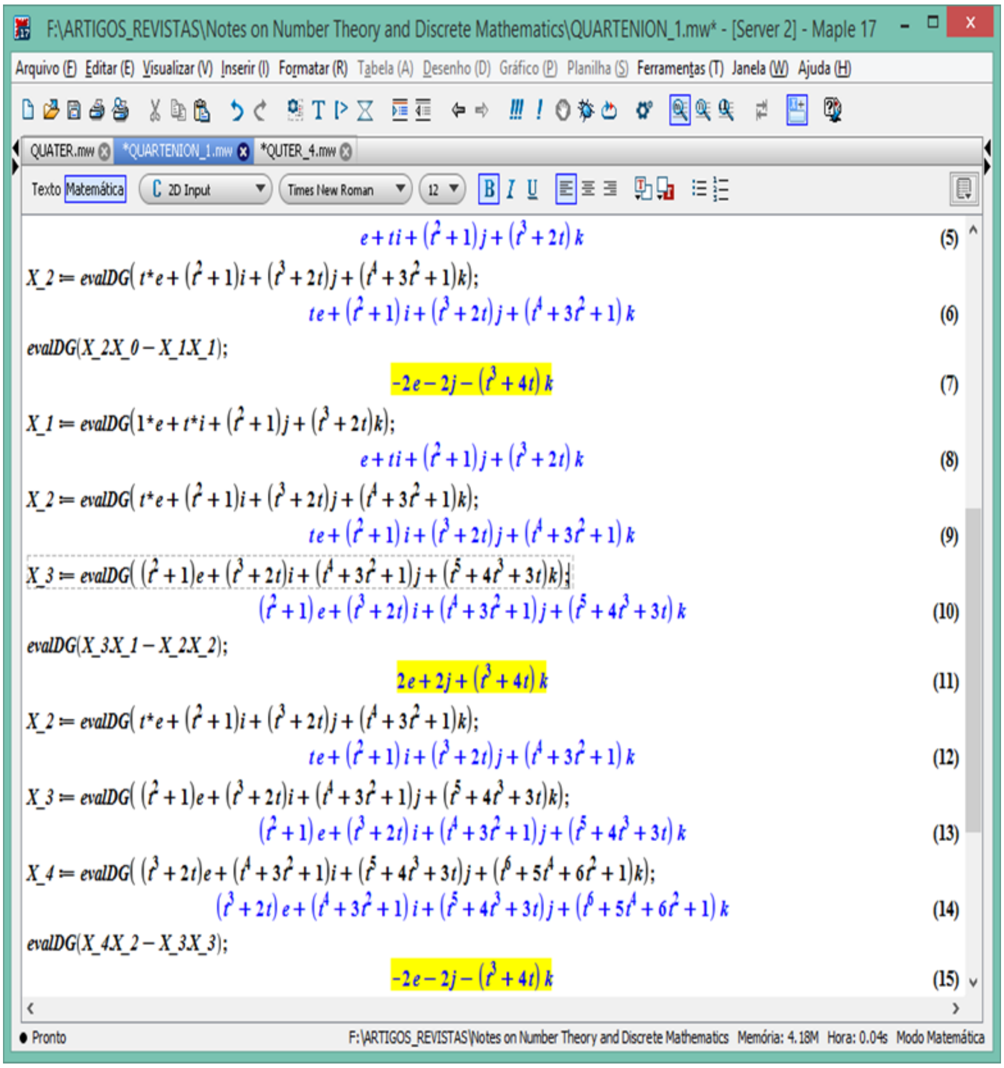


**Figure 7.** Checking the corresponding values for the k-Fibonacci quaternions

From the **Figure 8**, we consider the corresponding set of values relatively the k-octonions of Fibonacci, indicated by $Q_{k,n} = F_{k,n}e_1 + F_{k,n+1}e_2 + F_{k,n+1}e_3 + F_{k,n+2}e_4 + F_{k,n+3}e_5 + F_{k,n+4}e_6 + F_{k,n+5}e_7 + F_{k,n+6}e_8$. Similarly, we take the formula $Q_{k,-n} = F_{k,-n}e_1 + F_{k,-n+1}e_2 + F_{k,-n+1}e_3 + F_{k,-n+2}e_4 + F_{k,-n+3}e_5 + F_{k,-n+4}e_6 + F_{k,-n+5}e_7 + F_{k,-n+6}e_8$. From this, we always find the same expression (see **Figure 8**): $2e_1 + 2e_3 + 2ke_4 + (2k^2+2)e_5 + (2k^5 + 10k^3 + 8k)e_6 + (2k^6 + 10k^4 + 10k^2 + 2)e_7 + (k^7 + 6k^5 + 10k^3 + 6k)e_8$, where $t \in IR$. On the other hand, the last expression can be rewritten as indicated in the theorem 4.

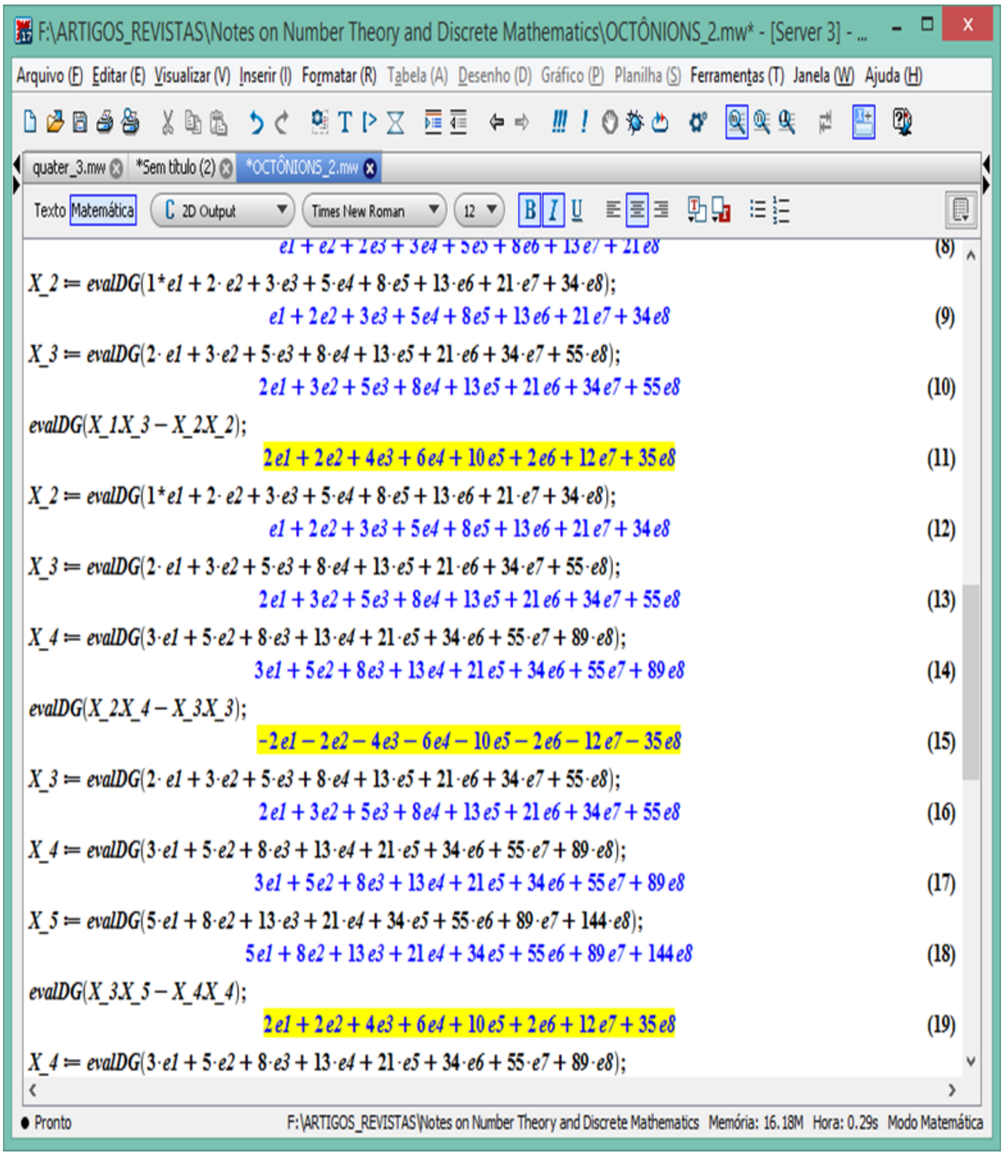


**Figure 8.** Checking the corresponding values for the Fibonacci octonions

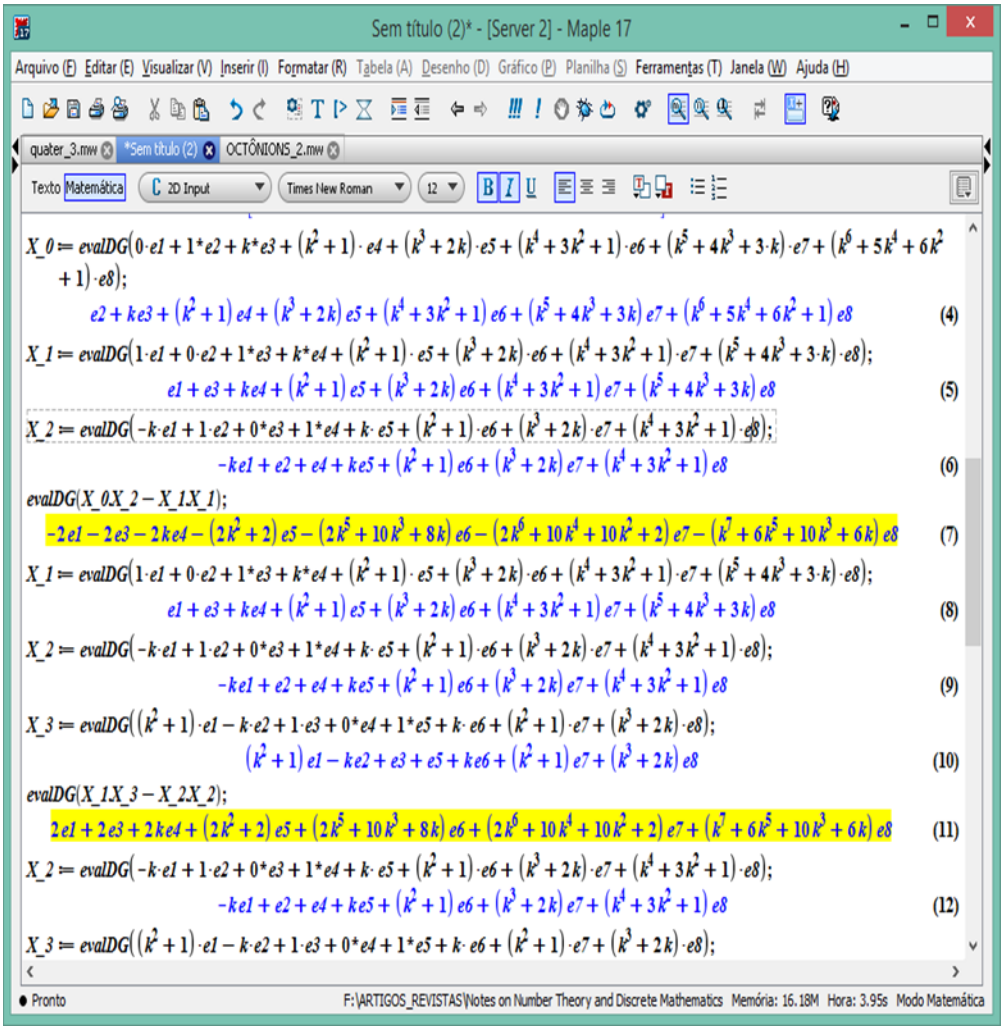


**Figure 9.** Checking the corresponding values for the k-Fibonacci octonions

To conclude this session, we noted the numerical invariance character in all situations, as we do the particular check for a considerable set of subscript interger indices. We recall that, even with a symbolic notational system of Fibonacci quaternions and Fibonacci octonions, the verification form of certain properties (and identities) may incur any mathematical errors present in some papers. For example, we have the case of the conjecture about Catalan's formula that proved wrong (Polatli & Kesim, 2015; Ramirez, 2015). Certainly, technological support can help us to produce conjectures with greater chances of success, given that, Mathematics does not progress only by the correct and precise arguments.

Moreover, with the use of technology we can see, conjecturing and, in some cases, make any corrections mathematical formulas. For example, in Catarino (2016, p. 74), we found the general identity $\left(Q_{\mathrm{p,q},n}(x)\right)^2 = 2f_{p,q,n}(x) - Q_{\mathrm{p,q},n}(x)\overline{Q_{\mathrm{p,q},n}(x)}$, where the terms $f_{p,q,n}(x)$ and $Q_{\mathrm{p,q},n}(x)$ are, respectively, the (p,q)-Fibonacci sequence and the (p,q)-Fibonacci quaternion. On the other hand, we make a correction and assume that $(Q_{p,q,n}(x))^2 = 2F_{p,q,n}(x)Q_{p,q,n}(x) - Q_{p,q,n}(x)\overline{Q_{p,q,n}(x)}$. From this, we can explore some particular cases with the Maple. In the particular case, we can observe, in the **Figure 10**, we can see the final result of the operation. From this, we can conjeture a we can suggest a correction in the formula indicated by Catarino (2016).

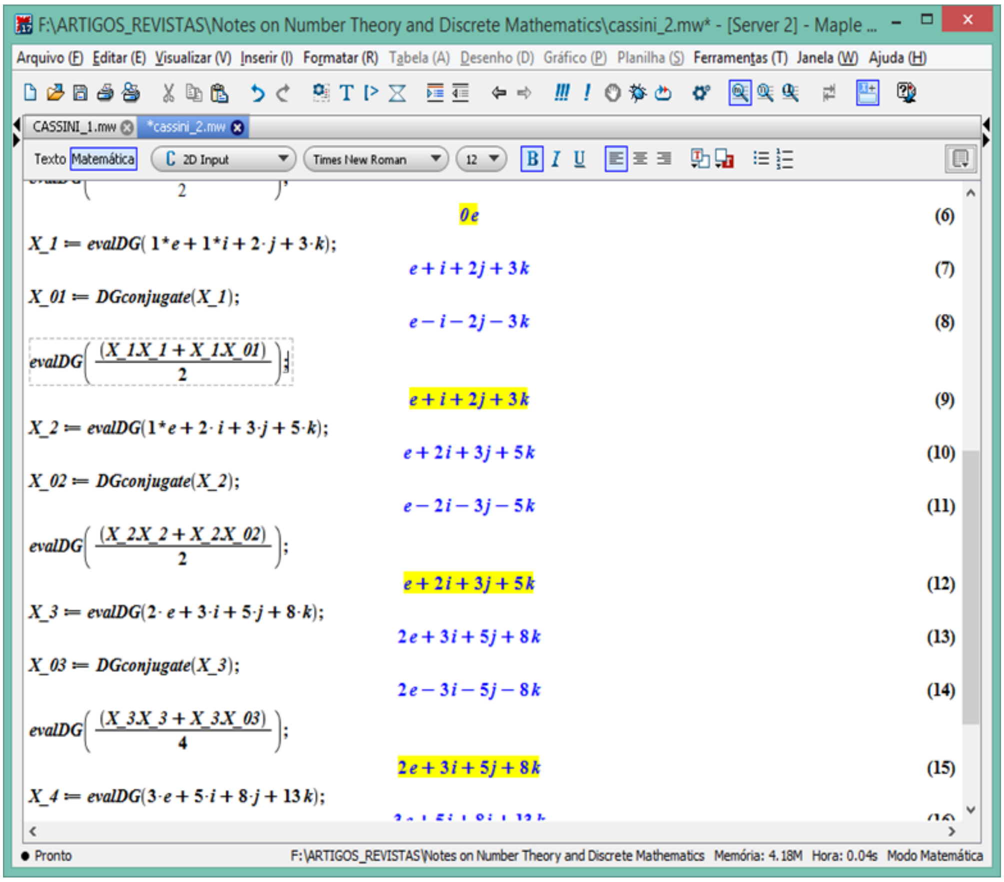


**Figure 10.** Correction of a algebraic identity relatively to the Fibonacci quaternion

**Table 1.** The behavior of the Cassini's identity for the Generalized Fibonacci Model for 'n' any integer number, and 'k' real number

| Derived forms of Fibonacci's sequence | The Cassini's generalized expression |
|---|---|
| Fibonacci sequence | $f_{n-1}f_{n+1} - f_n^2$ |
| k-Fibonacci sequence | $f_{k,n-1}f_{k,n+1} - f_{k,n}^2$ |
| Fibonacci quaternions | $Q_{n-1}Q_{n+1} - Q_n^2$ |
| $(-1)^n(2 + 2i + 4j + 3k)$ (see **Figure 6**) | |
| k- Fibonacci quaternions | $Q_{k,n-1}Q_{k,n+1} - Q_{k,n}^2$ |
| $(-1)^n(2 + 2j + (t^3 + 4t)k)$ (see **Figure 7**) | |
| Fibonacci octonions | $O_{n-1}O_{n+1} - O_n^2$ |
| $(-1)^n(2e_1 + 2e_2 + 4e_3 + 6e_4 + 10e_5 + 2e_6 + 12e_7 + 35e_8)$ (see **Figure 8**) | |
| k- Fibonacci octonions | $O_{k,n-1}O_{k,n+1} - O_{k,n}^2$ |
| $(-1)^n(2e_1 + 2e_3 + 2ke_4 + (2k^2 + 2)e_5 + (2k^5 + 10k^4 + 10k^2 + 2)e_7 + (k^7 + 6k^5 + 10k^3 + 6ke_8)$ (see **Figure 9**) | |

## CONCLUSIONS

In this work, we seek to emphasize some elements that can improve an historical investigation's conception. In a particular way, we discussed some properties derived from the Fibonacci's model, specially, some actual properties studied by several specialists. Moreover, with the Maple's help, we have analyzed, case by case, the final numerical behavior relatively the Cassini's generalized formulas, classically indicated by the expressions. From the invariant the expression behavior, we can make a statement by mathematical induction.

In this trajectory, we cannot disregard the role and the importance of formulation and establishment of new mathematical definition. In addition, from a set of four mathematical definitions, we have commented some rich properties derived from the Generalized Fibonacci Sequence. Thus, in the time interval of a few decades, we can understanding the evolutionary, unstoppable and epistemological progress of this conceptual and emblematic object. Furthermore, we enunciated some theorems extracted by the theory of Fibonacci quaternions and Fibonacci octonions that are discussed in a restrictive and encoded style, only in the mathematical scientific papers.

Regarding the Maple's use, we highlight the elements: (i) The software enables verifications of particular cases and properties related to the Fibonacci quaternions and Fibonacci octonions; (ii) The software allows the verification properties provided by classical theorems related to the Fibonacci quaternions and Fibonacci octonions, especially the most recently discussed in the literature; (iii) The software allows the description of a lot of special particular cases conditioned by newly formulated mathematical definitions; (iv) The software enables verification of properties related to a larger set of integer subscripts indicated in the scientific articles; (v) the software enables verification of a large number of individual cases in order to test mathematical conjectures; (vi) the software allows the correction of mathematical formulas in order to provide a correct description.

Finally, through the invariance numerical and algebraic of behavior, we can conjecture the intermediate steps of the inductive process. Thus, as in all verified cases, determine a simplified Formula for the Cassini's identity Casino and thus avoid certain mathematical errors that might be observed in other studies. Moreover, from the historical evolution of the quaternions and octonions' model, we can understand the current evolution of the research process around of the inherited Leonardo Pisano's model.

## Disclosure statement

No potential conflict of interest was reported by the authors.

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