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Rethinking mathematics teachers' professional knowledge for teaching probability from the perspective of probabilistic reasoning: A proposed framework

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ABSTRACT

Probability signifies a mainstream strand in mathematics curricula. Nonetheless, many curricular documents prepared for teachers might not offer enough support. In such a situation, a further reflection on teachers' professional knowledge for teaching probability is demanded; especially, from the perspective of probabilistic reasoning (PoPR) that is consistent with the need to pave the way for theories about mathematics education and cognitive psychology to consolidate achievements from each other. Accordingly, this study aims at conceptualizing mathematics teachers' professional knowledge for teaching probability from the PoPR. The initial step towards this conceptualization started by inferring the fundamental entities of teachers' professionalism through utilizing the mathematical knowledge for teaching model. Following this, three significant propositions were acknowledged. As a result, a conceptual framework was proposed, and a practical example was described. Such a description symbolizes a transition from emphasizing content knowledge towards highlighting teachers' process knowledge, which may impact the development of probability education research.

Keywords: probability, probabilistic reasoning, professional knowledge, mathematics teachers

INTRODUCTION

When the discussion lies in the arena of professionalism, the term professional knowledge always appears; expressly, with the growing importance of international comparative studies on learning outcomes (e.g., Trends in International Mathematics and Science Study [TIMSS], Programme for International Student Assessment [PISA]) wherein teachers' knowledge and its influence on the instructional quality and students' achievement has taken much attention. Furthermore, teachers' knowledge often defines as the heart of their professional competence (Ball et al., 2001; Shulman, 1986), which is well-described in the Teacher Education and Development Study: Learning to Teach Mathematics (TEDS-M) and the Cognitive Activation in the Mathematics Classroom and Professional Competence of Teachers (COACTIV) study (Baumert & Kunter, 2013; Kaiser et al., 2017; Kunter et al., 2013). However, both frameworks sharpen assessing mathematics teachers' knowledge generally. In other words, they neither conceptualize nor focus on probability. Notably, although research concerning teachers' knowledge for teaching mathematics is abundant, studies related, specifically, to probability are rare (Callingham & Watson, 2011; Torres et al., 2016).

The specificity of probability has been emphasized in several studies. For a case, Batanero et al. (2004) stated that broad statistical knowledge, even when essential, is not enough for teachers to teach probability. Although the probability is authorized in different stages from primary to teacher education curriculum, its inclusion in the curriculum does not automatically guarantee accurate teaching and learning. It has some specific characteristics, such as a multifaceted view and the lack of reversibility of random experiments, which are not usually encountered in other mathematics areas (Batanero et al., 2016). Consequently, several researchers reported that without specific training in probability, preservice and practicing teachers (and perhaps some teacher educators) may rely on their beliefs and share similar misconceptions with their students (Fischbein & Schnarch, 1997; Konold et al., 1993; Pratt, 2005; Prodromou, 2012; Shaughnessy, 1977; Stohl, 2005).

From this aspect and to address this arena under the umbrella of competence models' creation (Krainer & Llinares, 2010), the current study attempts to conceptualize mathematics teachers' professional knowledge for teaching probability from the perspective of probabilistic reasoning (PoPR). Although probability was unequally implemented in school and teacher education curricula while conducting the TEDS-M (Li & Wisenbaker, 2008), nowadays, growing attention is given to probability due to its relevance for applications in everyday life and sciences (Gal, 2005). This is recently declared by Hokor (2020) as probability defines one essential concept in our daily life through which better decisions could be performed, particularly in uncertain situations. Accordingly, probability as a content area has emerged globally as a mainstream strand in mathematics curricula (Jones et al., 2007) and was included in the National Council of Teachers of Mathematics (NCTM) standards from kindergarten to the secondary level (NCTM, 2000).

Acknowledging that the creation of such competence models in probability requires much focus on thinking processes while teachers still approach statistics and probability lessons like other mathematical topics; they focus only on procedures and results rather than thinking and reasoning processes (the 10th Congress of European Research in Mathematics Education [CERME10]) (Dooley & Gueudet, 2017). This is detailed in the Guidelines for Assessment and Instruction in Statistics Education's (GAISE) report since college students should learn statistical thinking to cook creatively instead of merely following traditional recipes (Franklin et al., 2007). Moreover, Garfield and Ben-Zvi (2008) asserted that the main challenge in teaching and learning statistics is to ensure that students have not only obtained the mechanics of statistical methods but, further, concepts underlying statistical reasoning.

The above argumentation is consistent with the direction of the future research (Chernoff & Sriraman, 2014), wherein the need to "pave the way for theories about mathematics education and cognitive psychology to recognize and incorporate achievements from the other domain of research" (Gillard et al., 2009, p. 13) has been acknowledged. That is significant to discuss within the scope of teachers' knowledge since psychologically, in the case of probability, the world of personal attitudes and intuitions signifies one source of success or failure of teaching and determines whether learners accept or ignore what they learn (Kapadia & Borovcnik, 2010).

THE FRAMEWORK PROGRESSION

In order to conceptualize mathematics teachers' professional knowledge for teaching probability from the perspective of probabilistic reasoning (PoPR), and further exhibit it through the expected framework, the following steps were conducted.

Step I: Reviewing the Literature on Teachers' Knowledge for Teaching Probability

In this step, the mathematical knowledge for teaching (MKT) model (Ball et al., 2008) was used to crystallize the previous studies on teachers' knowledge. It signifies a well-defined practice-based framework utilized by many organizations to drive the improvement of teaching (Kleickmann et al., 2013). According to the MKT, teachers' knowledge comprises subject matter knowledge (SMK) and pedagogical content knowledge (PCK) that involves knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC). Thus, this framework worked significantly as a lens to categorize the studies on teachers' knowledge for teaching probability through which the initial entities of the framework were determined as follows:

Knowledge of probability [KoP] (The essence of professional knowledge)

Since this study speculates the PoPR that implicitly includes teachers' conceptions, it is worthy to note Batanero et al. (2010), Godino et al. (2008), and Torres and Contreras's (2014) interpretation of teachers' professional knowledge for teaching probability. According to them, the epistemological reflection on the meaning of concepts, which corresponds to the notion of SMK within the MKT model, denotes an essential component.

Probability has two primary interpretations that are the statistical and epistemic facets. The statistical side is relevant to the objective mathematical rules that govern random processes. On the contrary, the complementary epistemic side views probability as a personal degree of belief; it depends on the information available to the person assigning the probability (Hacking, 1975). Accordingly, the theoretical, experimental, intuitive, and subjective probability imply the primary interpretations, in which they appear in the K-12 school curriculum, and mathematics teachers should understand it to teach probability for primary and lower-secondary school students (Dollard, 2011; Sharma, 2016; Torres & Contreras, 2014; Torres et al., 2016).

The theoretical and the experimental probability represent the two main interpretations in the objective school, in which none of them is suitable for all situations, and the appropriate approach of probability should be adapted depending upon the context (Kvatinsky & Even, 2002, 2010; Torres & Contreras, 2014). During classroom instruction, theoretical probability considers the most used interpretation. It can be easily applied to several random devices such as dices in which the outcomes of the sample space are assumed to be equally likely. Furthermore, it enables teachers to avoid the uncertainty of real random phenomena (Dollard, 2011; Stohl, 2005). On the other side, because of the increasing interest in using technology and simulation software in teaching probability, the experimental approach is now receiving growing treatment (Batanero et al., 2005b, 2016). That explicitly appears in many curricula standards documents such as the NCTM (2000) and the Common Core State Standards for Mathematics (CCSSM, 2010).

The subjective approach treats probability as a language for describing the level of uncertainty that one feels (Liberman & Tversky, 1996). Thus, different people may assign different subjective probabilities to the same event (e.g., election results) if they have different information or scope of view (Dollard, 2011; Kvatinsky & Even, 2002). Formally and by the school curriculum, this approach can be implemented through the intuitive explanation and the concept of conditional probability. The intuitive interpretation appears at the primary level when students first encounter the notion of probability through employing a variety of qualitative expressions (i.e., probable, unlikely, and possible) to express their degrees of confidence in the occurrence of events (Batanero et al., 2005a; Godino et al., 1987). Later, when the students enter secondary school, they learn the subjective probability through the concept of conditional probability and the Bayesian theorem (Lindley, 1994), which describes an update of the predictor's knowledge of a particular event when additional information is provided (Kvatinsky & Even, 2002).

Knowledge of teaching probability [KoTP]

There are many aspects relevant to teaching probability were emphasized in the literature, for example:

1. **Warm up the probability lesson:** Teachers should approach the probabilistic ideas through daily activities, which help students adapt their intuitive understanding of uncertainty to capture the formal concept of probability (Kataoka et al., 2008).
2. **Access activities of probability:** Teachers' knowledge to explain the probabilistic concepts appear in the textbooks represents a basic repertoire for teaching probability (Kvatinsky & Even, 2002). In other words, many concerns regarding teachers' capacity to access the implemented activities have been highlighted in the literature (Gusmão et al., 2010; Kataoka et al., 2010; Torres et al., 2016). For example, teachers should know how to calculate simple, compound, and conditional probability, understand the concepts of variability, expectation, randomness, and independence, explain the meaning of the assigned number to probability, and distinguish between mutually exclusive events and independent events. Moreover, they also are required to differentiate between the mathematical and the statistical problem (Franklin et al., 2007).
3. **Connect and differentiate among the different interpretations of probability:** On one side, Batanero et al. (2016) regarded the value of the explicit differentiation between the theoretical model of probability and the frequency data from reality in a way wherein students can model the real-life phenomena. On the other side, Chaput et al. (2011), Gusmão et al. (2010), Savard (2010), Theis and Savard (2010), and Torres and Contreras (2014) emphasized the connection between theoretical and experimental approaches to enhance students' probabilistic reasoning. Such a connection leads to debating the law of large numbers (Dollard, 2011; Kapadia & Borovcnik, 2010; Sharma, 2016; Stohl, 2005). Consequently, teachers' knowledge to select and adapt the appropriate activity relevant to each probability interpretation is essential. While data modeling helps the students acquire the experimental probability, the simulation using theoretical probabilistic models will provide them with data (Prodromou, 2012).
4. **Utilize various representations of probability:** Many researchers emphasized the importance of teachers' familiarity with diverse representations to provide students with a sufficient understanding of probability (Danisman & Tanisli, 2017; Even & Kvatinsky, 2010; Theis & Savard, 2010). For instance, teachers may use tables, pipe diagrams, area models, Venn Diagrams, or tree diagrams to clarify the probability concepts adequately. Furthermore, the simulation process is widely discussed in the literature, either through some concrete materials (e.g., spinners) or via computerized simulators. It helps sustain students' motivation and overcome their deterministic reasoning by comparing the observed outcomes with their prior predictions (Grenon et al., 2010; Kapadia & Borovcnik, 2010).

Knowledge of students' probability knowledge [KoSPK]

According to Danisman and Tanisli (2017), teachers' knowledge about students includes recognizing their prior knowledge; misconceptions that were also stressed by Stohl (2005) as teachers should perceive students' conceptions of probability; difficulties; and various levels of cognitive development. For instance, students' understanding of ratios, proportions, percentages, fractions, and rational number concepts related to probability is crucial to be investigated. This demands profound curriculum knowledge (i.e., horizontal and vertical knowledge) through which mathematics teachers connect what students learn in previous grades with the requirements to understand current probability concepts. For this matter, Batanero et al. (2016) guided mathematics teachers to be aware of research results that explain students' probabilistic reasoning and misconceptions; and further the appropriate instructional approaches that can help develop that reasoning.

Knowledge of Probability Language [KoPL]

The consideration of language has been strengthened not only in the case of probability but also for the whole statistics education, in which discussing the statistical content by the teacher who is conscious of the statistical words positively affects students' understanding (Otani et al., 2018). More specifically, about probability, many researchers highlighted the probability language as a fundamental aspect of teachers' knowledge (Batanero et al., 2016; Brijlall, 2014; Danisman &Tanisli, 2017; Dollard, 2011; Gal, 2005; Gusmão et al., 2010; Park City Mathematics Institute [PCMI], 2017; Torres & Contreras, 2014). Within this scope, Skoumpourdi and Kalavassis (2003) declared that utilizing probabilistic expressions and suitable vocabularies for the students draws a necessary condition for warming up the probability lesson. Hence, connecting the students' everyday intuitions of chance manifested in their natural language with the academic language of probability signifies a further challenge for mathematics teachers (Batanero et al., 2016; PCMI, 2017). Moreover, teachers' awareness of the differences between both languages is needed (Kazima, 2007; Paul & Hlanganipai, 2014; Nacarato & Grando, 2014). As noted, the usage of probabilistic words during formal instruction sometimes varies from how these words are practiced in everyday life (Sharma, 2016). Based on the literature review detailed above, mathematics teachers' professional knowledge for teaching probability consolidates **KoP**, which outlines the heart of teachers' knowledge and indicates their deep understanding of the subject (Shulman, 1986). Also, this **KoP** intersects knowledge of the language, teaching, and students, to construct **KoPL**, **KoTP**, and **KoSPK**, respectively, as portrayed in **Figure 1**.

According to **Figure 1**, the exhibited interplay among the four aspects of teachers' knowledge can overcome two reservations regarding the MKT model. While the first reservation denotes using the term PCK that did not appear as an appropriate name to identify the right side (i.e., the combination of KCS, KCT, and KCC) of the MKT framework (Hurrell, 2013), the second implies that interactions among knowledge domains were not displayed, which stands an obstacle towards categorizing teachers' knowledge (Marks, 1990). For clarification, the **KoP** component in **Figure 1** exemplified the core of teachers' professional knowledge instead of being a distinct aspect by itself. Additionally, such **KoP**, which resembles SMK in the MKT model, intersects knowledge of the language, teaching, and students, to define **KoPL**, **KoTP**, and **KoSPK**, respectively, as stated earlier.

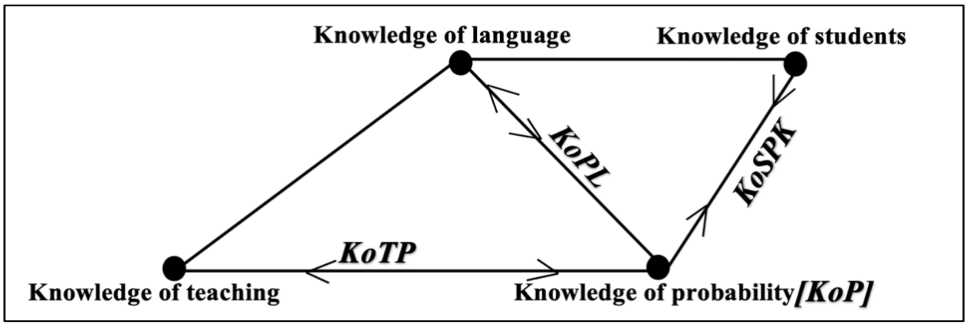

Figure 1. The initial entities of the proposed framework, building upon a literature review on probability education

The above paragraph explains how the interaction among aspects of teachers' knowledge was considered in the proposed model. Moreover, such interpretation acknowledges Shulman's (1987) original idea about the PCK that should embody the intersection between content knowledge and pedagogy (e.g., Marks, 1990). That is, PCK is a

> "special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (Shulman, 1987, p. 8),

which meets what Dewey (1964) declared that separating content from method distorts teachers' knowledge. This also matches Brijlall's (2014) exploration of PCK for teaching probability in the South African context, wherein a strong relationship between teachers' content knowledge and their teaching practices was exposed.

Since the MKT model signifies a distinctive contribution to addressing the demands for effective teaching that ensures a positive impact on students' performance (Kleickmann et al., 2013), several discussions regarding teachers' knowledge for teaching probability have adapted this perspective (Birel, 2017; Danisman & Tanisli, 2017; Kvatinsky & Even, 2002; Papaieronymou, 2009; Torres et al., 2016). For example, Danisman and Tanisli (2017) explored secondary school teachers' PCK of probability and revealed that their knowledge was insufficient; besides, teachers' beliefs were identified as the most influential factors affecting their PCK. Also, Chick and Baker (2005) detailed multiple issues about the content knowledge and PCK for two teachers who taught probability lessons to fifth-grade students. Hence, they highlighted that the probability concepts embedded in the curriculum and further appeared during the implementation should be understood by the teachers themselves.

Furthermore, for the pre-service mathematics teachers (PSMTs), Birel (2017) examined their SMK defined by procedural and conceptual knowledge of basic probability concepts. Accordingly, the results proved that although PSMTs showed a high achievement in procedural knowledge, most of them had difficulties solving questions that required conceptual knowledge. While Birel's (2017) study was conducted in the Turkish context, Contreras et al. (2011) confirmed the prospective Spanish primary school teachers' inadequate knowledge and the need to strengthen teachers' preparation to teach probability.

One critical point of these studies is that they centered around assessing teachers' practical knowledge and merely described it as insufficient or inadequate. More precisely, such investigations neither regarded teachers' reasoning processes nor the cognitive biases underpin their insufficient knowledge or practices. Perhaps that is because of the general tendency concerning teachers' knowledge research to strengthen the content knowledge more than process knowledge. Besides, the MKT approach cannot adequately satisfy the study of probability, wherein the probabilistic reasoning that varies from reasoning in other mathematics areas has a distinct emphasis (Stohl, 2005). As reported by Kapadia and Borovcnik (2010), to think probabilistically, when it is not always reasonable to seek a closed solution as one would expect in mathematics, it is time to substitute Heitele's (1975) fundamental ideas with an approach that perceives the concepts from a non-mathematical perspective. This pulls us back to the direction of future research that calls for connecting mathematics education perspective on probability with its roots in psychological research. Because of such concerns, the proposed framework has not only relied on issues raised in the previous studies (i.e., the first step), but it also attempted to express a new angle that may exhibit the psychological facet of teachers' knowledge for teaching probability represented by their reasoning processes and conceptions. These ideas are further detailed in the next step.

Step II: Defining the Study Propositions

In this step, three principal propositions were defined to exhibit the psychological facet of teachers' knowledge. These propositions outline the researcher's viewpoint on how the research gap can be fulfilled, and the findings of the first step can be complemented; thus, ultimately, the framework that defines mathematics teachers' professional knowledge for teaching probability from the PoPR was proposed. It includes:

Proposition 1: Conceptions are knowledge in evolution

Conception is knowledge produced by the interaction between an individual and the milieu (Gras & Totohasina, 1995). It is formed based on the individuals' personal experiences; hence, the conception signifies a mental filter to interpret a situation and make sense of it (Giordan & Pellaud, 2004). Furthermore, conception is not correct or incorrect, but rather it is operating or inefficiency (Giordan, 1998). Thus, Savard (2014) employed the term "alternative conception" to reflect the validity of a certain conception in some contexts and its inadequacy in others. In that sense, these conceptions represent knowledge in evolution.

Although the elementary probability is often determined through limited techniques, several deep conceptual issues (e.g., variation, randomness, fairness) stay essential to investigate (Chick & Baker, 2005). In this view, the complexity of probability conceptual understanding remains a fundamental obstacle while developing teachers' knowledge. Such complexity is originated from counterintuitive issues in probability; as reported by Borovcnik and Peard (1996), the counterintuitive results in probability are found even at very elementary levels, while they are encountered in other branches of mathematics when students work at a high degree of abstraction. These distinctive traits of probability explain why many conceptions and learning difficulties persist up to the university level (Batanero & Sánchez, 2005; Fischbein & Schnarch, 1997; Kapadia & Borovcnik, 1991; Konold et al., 1993; Stohl, 2005; Torres & Contreras, 2014).

Additionally, about the probability conceptual knowledge, it is valuable to note that such relevance of probability to daily life experiences has provoked a trend of research that recognizes socio-cultural influence on learners' conceptions of probability. For example, Amir and Williams (1999) concluded that students' cultural experiences impact their probability knowledge, in which some of them reveal superstitions of attributing random events to God. Similalry, Chassapis and Chatzivasileiou (2008) reported the influence of religious beliefs and social values on students' conceptions of probability, which confirms or contradicts mathematics education.

Moreover, to clarify the relationship among conceptions, conceptual understanding, and reasoning, it is not possible to pretend that a specific type of conceptions might exactly explain a certain level of understanding because classifying these conceptions as levels of conceptual understanding does not recognize the value of individuals' reasoning to make sense of phenomena (Savard, 2014). From this aspect, utilizing the PoPR would support admitting learners' various conceptions. This is crucial since individuals' world is full of diverging personal probabilistic conceptions (Kapadia & Borovcnik, 2010). Besides, these conceptions also signify a necessary component for the process of knowledge construction (Smith et al., 1993). Such studies acknowledge that students come to classrooms with previously formed beliefs and knowledge of probability (Fischbein, 1987). This is consistent with what Konold (1991) argued regarding students' construction of knowledge; the acquired knowledge is incorporated into their existing knowledge fabric. Notably, what students learn from the classroom experiences remains limited and is probably shaped by what they already know; accordingly, the acquired concepts are not freely formulated, but rather, they are subjected to restrictions of the existing concept-relations (Konold, 1991).

Proposition 2: Reasoning defines an individual cognitive process to interpret the acquired knowledge

Generally speaking, and from the perspective of teachers' knowledge, it is meaningful to note that across all teaching practices (e.g., figuring out what students know, manipulating representations, modifying textbooks), teachers' reasoning is always involved (Ball et al., 2001). Such argument stays significant for the probability instruction in which psychological interpretation feels at home (Van Dooren, 2014). Accordingly, and about teaching probability, Kapadia and Borovcnik (2010) regarded the time to replace Heitele's (1975) ideas, which resemble probability textbooks' chapters, with an approach that looks at concepts from a non-mathematical perspective, to overcome such distinct features of the probability teaching, wherein it is not always sensible to seek a closed solution as expected in mathematics. This non-mathematical perspective is displayed through this study as probabilistic reasoning, which has a cognitive psychological nature and focuses on how the mind works.

Since probability provides a distinct reasoning mode that contributes to developing students' mathematical reasoning (Batanero et al., 2016), probabilistic reasoning signifies one primary reason why probability stays embedded in the school curriculum (Borovcnik & Peard, 1996). This meets the demand to overcome individuals' deterministic thinking and admit the existence of chance in nature (Martignon, 2014). Furthermore, another critical issue for why probabilistic reasoning is appreciated in this study is the duality of the probability concept, which has statistical and subjective facets (Carranza & Kuzniak, 2008; Hacking, 1975).

In this regard, the conventional approach to addressing teachers' knowledge, which focuses on leveling their conceptual understanding, may remain unsuitable to employ because of the subjectivity; it depicts one plausible approach to interpreting probabilistic situations. This is well described by Brase et al. (2014), in which

> "having two different conceptions of probability can lead to two people having different answers to the same question yet both believing they are rational and correct" (p. 162).

Besides, strengthening the statistical facet, which reduces teaching probability to formula-based computational procedures with few models of real applications, as a unique basis to judge a probabilistic phenomenon deepens the gap between both facets (Carranza & Kuzniak, 2008).

The beforehand argumentation exposes the significance of the PoPR to conceptualize mathematics teachers' knowledge for teaching probability. It admits their conceptions and cognitive biases that should not be ignored; particularly, if they are not objectively acceptable, they must be eliminated, and alternative representations must be developed instead (Fischbein & Gazit, 1984). In other words, because intuitions about probability could impede its learning, it is crucial to investigate learners' reasoning and biases (Chiesi & Primi, 2009), which explains Sharma's (2016) recommendation of grounding the instruction in experiences that help learners overcome their misconceptions and develop an understanding based on probabilistic reasoning.

Probabilistic reasoning implies judgments and decision-making under uncertainty (Falk & Konold, 1992); it considers two concepts of variability and randomness (Chick & Baker, 2005). Variability locates at the heart of statistics, and it designates why it is so difficult to make decisions under uncertainty (Garfield & Ben-Zvi, 2005; Pfannkuch & Wild, 2004). Randomness includes uncertainty and independence; while the former reflects that the outcome cannot be predicted definitely, the latter indicates no correlation between what happened before and the new outcome (Green, 1993; Savard, 2014).

Figure 2. Essentials to characterize mathematics teachers' knowledge for teaching probability

Accordingly, probabilistic reasoning differs from deterministic reasoning that (i) leads to looking at one definitive answer and (ii) seeks for a correlation using present and past information to explain a phenomenon, where the dependency or causality still exists (Savard, 2010, 2014; Shaughnessy, 1992). On the contrary, in a probabilistic situation, (i) there is more than one possible outcome, (ii) the occurrence of an exact outcome is unpredictable, and (iii) the sequence of obtained results lacks a pattern; it cannot be controlled or predicted, and the only thing to be done is to critically choose the event most likely to occur (Tsakiridou & Vavyla, 2015). This satisfies Borovcnik and Peard's (1996) differentiation between logical reasoning that designates a proposition as valid or false and probabilistic reasoning where we have no complete certitude concerning a random event in the case of probability.

Proposition 3: The hypothetical relationship between conceptions and reasoning

Considering the previously described propositions, the relationship between the individuals' probabilistic reasoning and their conceptions was interpreted in this study as follows: Depending upon the way we reason in an uncertain situation that contains probability knowledge (theoretical constructs), our conceptions could be uncovered. Some researchers implicitly declared such a connection by stating that probability conceptions are rooted in various epistemologies while those epistemologies themselves are underlined by the reasoning employed to think about probabilistic phenomena. For example, Konold (1989) noted that reasoning about uncertainty involves two types of cognition: formal knowledge of probability and intuitive assessments (heuristics). Later, these types were redefined by Savard (2014) as probabilistic vs. deterministic reasoning. Indeed, admitting such a relationship does not only support defining the framework but also contributes to the literature by consolidating teachers' reasoning and probability conceptions together in one model. Although several studies showed that adults hold various conceptions about probability and relevant biases in reasoning under uncertainty (e.g., Dollard, 2011; Kazak & Pratt, 2017; Konold, 1989), there is no further discussion that connects teachers' reasoning with associated probabilistic conceptions in such a way to prototype both in a unified schema. From this aspect, this study acknowledges that learners' conceptions are underlined by their way of reasoning toward a certain phenomenon to be an essential hypothesis. In other words, one way to identify teachers' conceptions of probability are to explore how they reason under uncertainty.

Lastly, and after reviewing several studies in both fields of cognitive psychology and mathematics education (e.g., Batanero & Sánchez, 2005; Díaz & Batanero, 2009; Díaz & de la Fuente, 2007; Dollard, 2011; Garfield & Ben-Zvi, 2005; Kazak & Pratt, 2017; Konold, 1989; Lysoe, 2008; Savard, 2014; Tversky & Kahneman, 1974), **Figure 2** was developed. It determines (i) characteristics of the individual who holds a conception (misconception, heuristic, or bias) and (ii) the principal probability interpretations that the

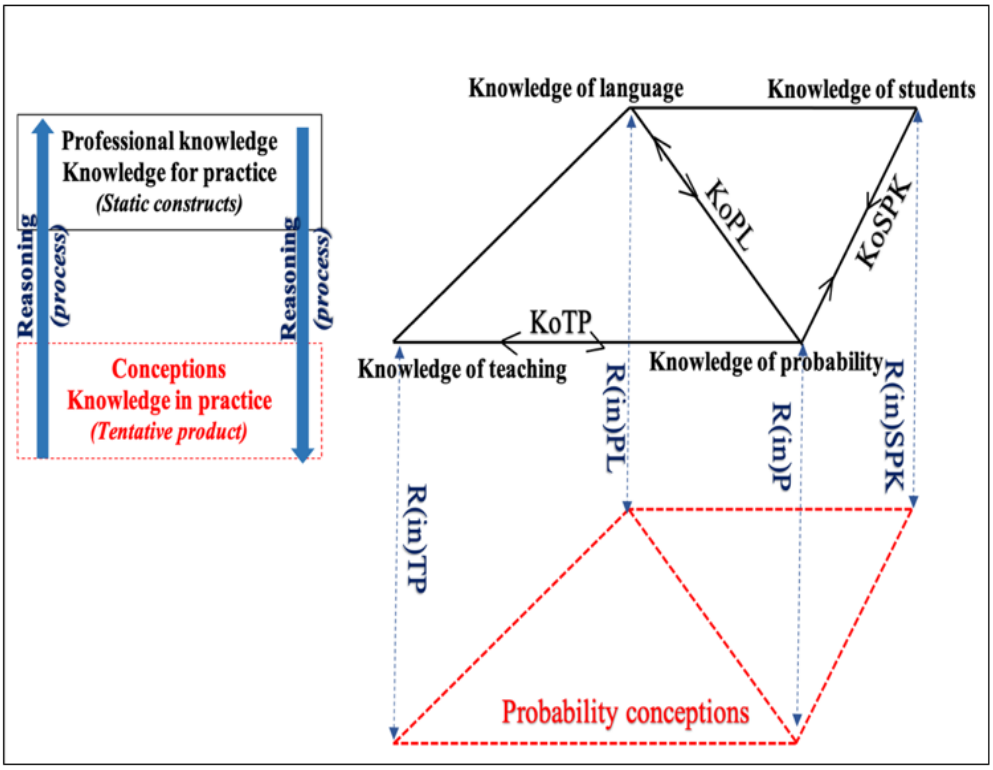

Figure 3. The framework of mathematics teachers' professional knowledge for teaching probability from the perspective of probabilistic reasoning

individual might rely on to reason in a situation, which worked as a lens to interpret some responses of which the framework has been validated practically.

Theoretical Description of the Framework

In light of the preceding discussion that (i) determined the initial entities of the framework and (ii) acknowledged the study propositions, the study framework is displayed in **Figure 3**. It defines mathematics teachers' professional knowledge for teaching probability from the PoPR, which embodies interrelationships among professional knowledge, conceptions, and reasoning processes.

According to the presented model, mathematics teachers' professional knowledge for teaching probability (knowledge for practice), acquired through either formal teacher education or professional development training, signifies the static black parallelogram. It consolidates **KoP** that outlines the essence of this parallelogram, which crosses with knowledge of the language, teaching, and students to assemble **KoPL**, **KoTP**, and **KoSPK**, respectively. Nonetheless, practically during the actual teaching, each teacher transmits this knowledge through his/her own lens; it indicates probability conceptions represented by the red parallelogram. This red parallelogram describes teachers' practical knowledge (knowledge in practice); it could match the black parallelogram when teachers' conceptions agree with scientific knowledge (theoretical static constructs). Still, there is a gap between how a teacher perceives (then implements) probability knowledge and professional knowledge for teaching probability if his/her conceptions do not fully fulfill the probability theory.

The existence of this gap reflects teachers' various ways of reasoning under uncertainty. That is, when a teacher operates his/her own reasoning in a situation that contains standardized probability knowledge (i.e., **KoP**, **KoPL**, **KoTP**, and **KoSPK**), he/she develops a particular distinct type of knowledge (i.e., knowledge in practice, knowledge in evolution, teachers' conceptions of concepts embedded in an instructional activity). This way, placing the focus on reasoning processes helps characterize such similar gaps. Alternatively, acknowledging the PoPR may respond to what was raised regarding the needed research that bases probability instruction (the perspective of mathematics education) on its psychological roots. Concretely, it (i) manifests the influence of teachers' reasoning under uncertainty in shaping their probability knowledge (conceptions) and (ii) reflects the possibly existing distance between these conceptions and what the educational community recommends mathematics teachers comprehend for teaching probability. Accordingly, effective instructional interventions can be organized to minimize such a distance.

It is also worthy to perceive that the described link between knowledge and conceptions does not convey a linear relationship that always begins with knowledge. Instead, the opposite direction still works since such resultant conceptions will not be isolated but integrated into a complex system (knowledge system). In other words, new knowledge does not destroy current knowledge; rather, it will be connected to existing concepts to reorganize and keep the individual's cognitive structure balanced (Savard, 2014; Vosniadou & Verschaffel, 2004).

From this aspect, and through the lens of probabilistic reasoning, mathematics teachers' professional knowledge for teaching probability includes these redefined aspects: **R(in)P**, **R(in)PL**, **R(in)TP**, and **R(in)SPK**, which symbolize their reasoning in a

Table 1. PSMTs' manners of reasoning in the context of weather predictability

Mathematically oriented [M]	Subjectively oriented [S]		Outcome oriented [O]		Intuitively oriented [I]
	Type s*	Type s**	Type o*	Type o**	
It means that probability of no rain equals 40%; or if rainy, windy, sunny, etc. are possible outcomes, then probability of all these outcomes (except rain) equals 40%.	60% probability of rain indicates a 40% probability of no rain; still, such probability is a matter of Allah's will.	60% probability of rain does not reflect an absolute value. Probability depends on many factors such as season, inclination, intensity of clouds, & wind movement.	It may rain tomorrow because it is winter, sky is dense, weather is cloudy, or it was announced on weather forecast.	A 60% chance of rain has been calculated on similar prior circumstances.	It is most probable that it will rain tomorrow, as 60%>50%.
7	**1**	**4**	**20**	**4**	**12**
7 responses	**5 responses**		**24 responses**		**12 responses**

situation that involves knowledge of probability, probability language, teaching probability, and students' probability knowledge, respectively (see **Figure 3**).

Preliminary Validation of the Hypothetical Framework

This section focuses on clarifying the aspect of **R(in)P,** practically, through a case of 48 PSMTs. They represent the available sample (Lopez & Whitehead, 2013) who study the mathematics teachers' preparation program at the Faculty of Education, Tanta University, Egypt.

The mathematics teacher preparation program in the Faculty of Education consists of four years. All PSMTs have to study advanced mathematics during the first two years; later on, during the last two years, the focus of this program will be slightly shifted towards learning the educational courses, including the curriculum and instruction of school mathematics (in addition to the advanced mathematics courses). The study participants were the student teachers who registered in the second, third, and fourth years of this preparation program in 2018/2019. They all had prior knowledge about theoretical, experimental, and conditional probability, either in secondary school or during their teacher education (see more details in Elbehary, 2021). Furthermore, the context of weather predictability was adapted upon Konold's (1989) study to approach PSMTs' **R(in)P** because it is familiar to the participant, and its classification as an intermediate level of difficulty when reasoning under uncertainty (Konold, 1989). Accordingly, they were asked to **explain the meaning of this statement[1]: the probability of raining tomorrow equals 60%.** Later, their responses were coded and categorized by utilizing NVivo software and Thomas's (2006) steps. Accordingly, four distinct categories emerged as follows (**Table 1**).

The first resultant category is **M**, wherein seven respondents modeled the problem of weather predictability as if the sample space contained two mutually exclusive events of rain and no rain. Consequently, a 60% chance of rain reflects 40% of no rain (the complementary event). That is, mathematically speaking, S={rain, no rain}, A={rain}, and P(A)=60%; then, P(Ac)=40%. Furthermore, a teacher in this category interpreted a 60% chance of rain as if there were various possible outcomes regarding tomorrow's weather, such as rainy, windy, and sunny. Accordingly, if the probability of rain tomorrow equals 60%, this means that the sum of all other possible outcomes equals 40%. Alternatively, S={rainy, windy, sunny, etc.} and P(A=rain)= 60%; then, P(Ac=wind+sunny+etc.)=40%. Such reasoning indicates utilizing **theoretical probability** since this 60% was decided based on various plausible events in the sample space.

All the **M** category respondents expressed attention to the **context** (i.e., actual circumstances). Moreover, they understood the concept of **variability**, which reflects that the resultant outcome varies depending upon the possible events in the sample space. On the other hand, they shared the **equiprobable bias** (Lecoutre, 1992), which formed their conception of **randomness**. This appeared when a teacher listed the outcomes of rainy, windy, sunny, etc., in one group to represent the sample space; he seemingly considered that all these events are equally likely to occur, which contradicts the given data regarding 60% of rain.

The second category of PSMTs' reasoning is **S**, which included five responses; while one answer was coded under **s***, four others were assigned to **s****. In **s***, a teacher persisted in manifesting the concept of **Allah's will** to reflect the uncertainty of the rain falling. He commented,

> "first, a 60% chance of rain reflects a 40% chance of no rain; yet we cannot expect rain to occur because the actual event may alter depending upon Allah's will."

This reasoning reflects the usage of Allah's will concept not as a cause to explain the variability but rather as a factor that may interfere with the situation.

Also, within the main category of **S**, four teachers were assigned to **s****. For them, a 60% chance of rain did not indicate a certain percentage; instead, it defined their various degrees of uncertainty regarding weather conditions (Liberman & Tversky, 1996). They reported:

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[1] Two more questions were presented to participants in addition to the mentioned one (see Elbehary, 2021); however, considering this paper limitations, merely results of this item were sharpened.

"a 60% chance of rain may designate several circumstances, such as the temperature, season (winter or summer), movements and intensity of clouds, or flow inclination; this percentage was judged in light of all these circumstances."

Hence, for **s**,** the given probability did not specify one issue of only clouds (as a case) but rather many factors that worked together to determine that probability. Hence, the prediction may vary upon what we know about all these criteria.

About the **context**, all **S** showed the **data context** (Pfannkuch, 2011), wherein they all relied on several actual conditions to explain the situation of weather predictability. Furthermore, the **variability** for **S** indicated that the expected outcome alters depending upon the available conditions regarding the phenomenon under study. Nonetheless, the nature of these conditions varied between **s*** and **s****. Although **s**** attributed the variability to several cognitive criteria (i.e., environmental circumstances), **s*** emphasized the religious conception of Allah's will as a possible factor that may alter the outcome. Additionally, both **s*** and **s**** acknowledged the **randomness**. This emerged when they claimed that a 60% chance of rain was not an exact judgment, but it might alter depending upon the interplay among several conditions. Because **s*** and **s**** relied on the **subjective probability** to explain the given problem, it is reasonable that their conception of randomness was also subjective. It is no longer an objective physical property but rather a subjective judgment (Batanero, 2015).

The third category is **O**; it included **o*** and **o**** as two minor subcategories and represented 50% of all responses. Type **o*** reasoning reveals a partial understanding of **the experimental probability** in which teachers focused on the outcome itself rather than its probability. As shown in **Table 1**, 20 teachers interpreted a 60% chance of rain based on several causes due to which rainfall occurs. In other words, they based their predictions on *causal analysis* of the situation and sharpened the favorable outcome (i.e., rain occurrence) as if it had already occurred, and they were discussing its causes (under what circumstances did the rain occur?). For example, a 60% probability of rain reflects the humidity (12 responses), the current season (2 responses), the climate is cold or stormy with dust (3 responses), or the forecast announcement (3 responses). This case resembles what Konold (1989) reported regarding students who thought that humidity or cloudiness defines a measure of the strength of factors that would produce rain.

Such reasoning defeats the theory of **experimental probability** that reflects the limit of relative frequencies of an event when an experiment is repeated many times (Konold, 1989). Moreover, because **o*** respondents focused on the favorable outcome, which affected their utilization of the experimental approach, they lacked an understanding of the concept of **randomness** that requires **independence**. Alternatively, they shared the **causal conception** that made them **confuse causality with conditionality** since they supposed that the conditioning event (e.g., humidity or fuzzy sky) remains the cause, while the favorable outcome (i.e., rain occurrence) signifies the consequence. Additionally, although **o**** respondents resembled **o*** in manipulating **experimental probability,** they clearly understood it without sharing any biases or conceptions. Thus, four teachers clarified that a 60% probability of rain means

"in the past 100 days that had similar weather and environmental circumstances, rainfall occurred 60 times."

Such an explanation speculates based on adequate knowledge regarding the appropriate context in which the experimental probability can be operated. This matches Brase et al.'s (2014) determination of a 30% chance of rain; it describes a model of past weather events in which it rained on three out of the ten previous days that had similar circumstances.

Besides recognizing the real-world conditions from which the problem arose, **o**** showed another type of **contextual recognition:** the **task context** (Pfannkuch, 2011). That is, their reliance on the experimental probability to approach the weather predictability problem designates a clear understanding of the appropriate circumstances when that experimental probability works. Regarding **variability**, although all **O** admitted it, how they perceived such variability was quite different. For **o*, variability** did not depend on the frequencies; instead, on **one single trial** through which the favorable outcome can be interpreted. This explains why they adjusted their expectations to be within two sets: one contained the favorable outcome, while the other included all other outcomes (i.e., the complementary set). On the contrary, **o**** exposed sufficient knowledge of the variability in which the estimation varies depending upon the frequencies in the total number of performed trials. Additionally, about the **randomness**, as reported earlier, **o*** respondents exposed the **causal conception**. It denies the **independence** that remains an essential feature of probabilistic reasoning. Again, **o**** respondents displayed a tacit recognition of **randomness**, which could generate a fair distribution in the long term if and only if the number of trials was increased.

The last emerged category is **I**; it incorporated 12 teachers who transformed the quantitative expression of a 60% chance of rain into the qualitative one:

"It does mean that: It is most probable that it will rain tomorrow."

Furthermore, while one teacher continued his answer by declaring,

"Because 60% is higher than 50**%,** I reported that: It is most probable that it will rain,"

all other teachers wrote,

"Still, we are not sure whether it is going to rain or not."

Such qualitative expressions speculate a novice understanding of the probability that reflects an encapsulation of intuitive views of chance and leads to idea of committing numbers to uncertain events, which implies **intuitive probability interpretation.**

Table 2. A model of PSMTs' **R(in)P** that is related to a simple probabilistic situation

Reasoning types	**Knowledge for practice (process knowledge)**		
	Variability	**Randomness**	**Contextual recognition**
M reasoning: It models the uncertain situation through the theoretical probability			
m	The outcomes vary depending upon several possible events in the sample space. **[Variability by sample space elements]**	Equiprobability (the random nature of the experiment remains a sufficient indication of equiprobable outcomes) **[Randomness as Equiprobability]**	Representativeness heuristic **[Context defines the realistic circumstances that are equal in occurrence to explain the uncertain situation]**
O reasoning: It models the uncertain situation through the experimental probability			
o*	The outcomes vary between two alternatives of either the favorable outcome or other remaining events. **[Variability by either a specific outcome or its complementary]**	Causal conception (the conditioning event is the cause of the favorable outcome occurrence) **[Randomness does not always require independence]**	Several daily life situations resemble the given contexts. **[Context defines realistic circumstances that resemble the uncertain situation]**
o**	The outcomes vary depending upon the possible resulting frequencies in a large number of performed trials. **[Variability by the frequencies of many trials]**	Randomness generates a fair distribution in the long term if and only if the number of trials has been increased. **[Randomness as stability of frequencies]**	Depending upon the circumstances of the uncertain phenomenon, the appropriate probability interpretation should be utilized. **[Context defines the conditions of the task that may sustain/hinder the utilization of a specific probability interpretation]**
S reasoning: It models the uncertain situation through the subjective probability			
s*	The outcomes vary depending upon Allah's will. **[Variability by Allah's will]**	Randomness still exists even after adapting the new information; that is, whatever we knew, the outcome could not be certainly anticipated. **[Randomness as a self-criterion based on the credibility of the available information]**	Several real situations may explain the probabilistic phenomenon. **[Context defines realistic circumstances that are known at the moment by a specific person to explain the uncertain situation; thus, the context is restricted by the information available to the person who judges the situation.**
s**	The outcomes vary depending on the multiple available information about the phenomena. **[Variability by the available information]**		
I reasoning: It explains the uncertain situation using qualitative expressions			
	The outcomes vary between two alternatives of either the favorable outcome or any other event. **[Variability by either a specific outcome or any other event]**	Randomness reflects any percentage that lies on the continuous decision line ranging from 0% to 100%. **[Randomness as an expression of any percentage ranging from 0 to 100]**	**[Context defines the usage of qualitative expressions to explain the uncertain situation]**

Although some researchers classified the **intuitive interpretation** under the subjective facet since the usage of qualitative idioms expresses the degree of individuals' confidence in the occurrence of an event (e.g., Torres & Contreras, 2014), **I** was not labeled as a sub-category of **S**. Instead, it was considered a category by itself. The reason for so is that when **I** respondents gave the expression of most probable, they judged it compared with 50%, whether the given percentage was higher or lower than 50%. Thus, according to them, the **variability** of outcomes (rain or no rain) did not speculate a subjective criterion but rather a mathematical standard. Beyond that, **I** thinkers understood the idea of **randomness**; it appeared in nearly all replies in which the uncertainty adequately resembled when they reported,

> "Still, because of 60% probability, we are not sure that it is going to rain."

Notwithstanding, **I** thinkers showed a novice recognition of the **task context** wherein the uncertain situation was explained qualitatively (Lysoe, 2008). Based on these results, the matrix validating the theoretical framework was developed (**Table 2**).

CONCLUSION

Since this study aimed at developing a framework of mathematics teachers' professional knowledge for teaching probability from the PoPR, it may be significant, especially for teacher education. It helps to clarify whether they accept (or ignore) what they learned about the formal theory of probability (Kapadia & Borovcnik, 2010). Furthermore, the detailed results stimulate PSMTs' awareness of probability conceptions, which, on the one hand, helps them assess these misconceptions later in their students (Batanero et al., 2010); and, on the other hand, works as a foundation to reform their pedagogical preparation. Ultimately, the study impacts pupils' reasoning wherein "the success of any probability curriculum for developing students' probabilistic reasoning depends greatly on teachers' understanding of probability" (Stohl, 2005, p. 345).

Despite that, some limitations should be considered regarding interpreting such results; firstly, and importantly, the study sample. The proposed framework has been validated through responses of a small sample of student teachers during their preparation program; in that sense, the results may change if a sample of in-service mathematics teachers were engaged in this study. Additionally, the question (item) used to validate the theoretical framework might represent another limitation; especially since the difficulty of reasoning in a probabilistic situation depends on the (i) sample space clarity, (ii) apparent chance factors, and (iii) cultural prescription toward viewing the phenomena statistically (Nisbett et al., 1983).

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