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Pre-Service Teachers' Understanding of Continuity

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ABSTRACT

In this paper, we report the analysis of thought processes used by Pre-Service Teachers' (PSTs') through clinical interviews as they solved an algebra task involving a linear pattern. The PST's were asked about a mathematical model they had constructed to describe a pattern problem. Our analysis suggests that conflict factors arise due to incompatibility in participants' personal concept definition and the formal concept definition of continuity. We identified how personal concept definitions of the participants differed and how this difference affected their decision on whether a graph was continuous or not.

Keywords: pre-service teachers, concept definition, concept image, conflict factors, linear pattern

INTRODUCTION

Common Core State Standards in Mathematics (CCSSM, 2010) lead us to examine teacher education at colleges and universities in the United States. The research reveals that pre-service teachers (PSTs) often do not have adequate subject matter knowledge as well as mathematical knowledge for teaching when they start teaching (Santagata & Lee, 2021). Hoover et al. (2016) assert that this mathematical knowledge involves an explicit conceptual understanding of the principles and the meaning underlying mathematical procedures, and connectedness rather than compartmentalization of mathematical topics and definitions. She states that improvements in academic and professional educational coursework should be a focus of current educational reforms.

For the development of sound and effective teaching techniques, it is important for curricula to establish connections between advanced mathematics courses that PSTs take and the courses they will teach in K-12. This is emphasized in the Committee on the Undergraduate Program Mathematics Curriculum (CUPM) Guides:

> The assumption that the traditional curriculum for a mathematics major is adequate preparation for students preparing to teach secondary school is simply incorrect. Having an understanding of advanced mathematics may not be enough for secondary mathematics teachers —they must also be able to connect their advanced coursework to the material they will teach (CUPM, 2004, p. 53).

The recent CUPM Guide (2015) also emphasizes the need to improve "ways to effectively connect the undergraduate mathematics major courses to school mathematics". In this study, we seek to examine whether, as suggested in the quote above, the traditional curriculum is adequate preparation for students preparing to teach secondary school. Our goal was to investigate PSTs' thinking process, as they attempted to solve the algebra task involving a linear pattern. We have made inferences about their thinking based on the clinical interview results. These inferences were about their personal concept definition and concept image of continuity (defined in the next section). Our study also shed light on the conflicts between students' personal concept definition and formal concept definition of continuity.

LITERATURE REVIEW, DEFINITIONS, AND FRAMEWORK

Tall and Vinner (1981) posited a framework that uses four ideas to explain what it means to understand a concept. These ideas are as follows: Concept image, formal concept definition, personal concept definition and conflict factors. This framework widely used in the research of mathematics education (Tirosh & Tsamir, 2021).

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Concept images are all mental images and associated properties and processes with concept names (Tall & Vinner, 1981). A student's concept image is the result of the student's familiarity and experience with that concept. In the concept of continuity, students can have an image of a graph which does not end anywhere.

The formal concept definition is the text book definition of the concept. For example, the concept definition for continuity can be "A function *f* is continuous at a number *a* if $\lim_{x \to a} f(x) = f(a)$" (Stewart, 2016). Limit is defined by Stewart (2016) as "Suppose *f*(*x*) is defined when *x* is near the number *a* (This means that *f* is defined on some open interval that contains *a*, except possibly at *a* itself.), then $\lim_{x \to a} f(x) = L$." (Tall & Bakar, 1992).

One's own concept definition does not have to be the same as the formal concept definition. Students may derive their personal concept definitions from their concept images and the formal concept definition. Therefore, their construction of personal concept definitions can happen in different ways. In some cases, they recite a formal concept definition without understanding it while in other cases they might transfer some part of the formal definition or an incorrect interpretation of the definition. A personal concept definition of continuity might be "no gaps in the graph". A student who has already constructed a concept image may not feel the need to check the formal concept definition. Most of the time students rely on their personal concept images and do not consult the formal concept definition to determine whether something is an example or non-example of a given mathematical object. Until they have more experience in mathematics, they do not need to use formal definitions (Tall & Bakar, 1992). There are also studies revealing that, in advanced continuity problems, students need to consult a formal definition of continuity in order to arrive at a correct and reasonable answer (Haciomeroglu & Schoen, 2008; Tall & Vinner, 1981). Conflict factors can exist if some part of the concept image or concept definition conflicts with another part of the concept definition. We speak of cognitive conflict in such cases (Tall & Vinner, 1981). When cognitive conflict happens (occurrence of disequilibrium), students are motivated to seek for new equilibrium state. Teachers play an important role at that stage to improve students' understanding by scaffolding the content (Susilawati et al., 2017).

METHODS

Pre-service teachers were asked to formulate a linear pattern based on a problem taken from Professional development guidebook for perspectives on the Teaching of Mathematics (Arcavi et al., 2016). Researchers then asked the participants about their solutions in one-on-one interviews. Interviews were recorded and PST's understanding of continuity was analyzed.

Participants and Data Collection

The participants were seven PSTs (one male and six females) enrolled into a methods course at a state university in the United States. This was the second methods course in the sequence that PSTs needed to take. Therefore, all the participants had completed at least one methods course prior to taking this course. In their methods courses, PSTs discuss the definitions, examples, and different representations of the topics from high school mathematics. The participants were chosen on a voluntary basis.

Interviews were conducted with the seven PSTs toward the end of the semester. During the interviews, PSTs were presented with an algebra task and asked to think aloud while they were solving the task so that we could analyze their responses and strategies as well as describe and make inferences about their concept images and concept definitions of continuity. After they were presented with the algebra task (**Figure 1**), they were given time to think about the pattern. To not distract other students, the individual interviews were conducted outside of the classroom. The individual interviews did not take more than 30 minutes and were audio-taped. Out of the seven PSTs, we focus on four of PSTs who had an incomplete understanding in the concept of continuity based on the explanations they provided. The study presented here is a multiple-case study.

Description of the Task

Figure 1 displays the algebra task, which involves generalizing a linear pattern and the PSTs were asked to "Write a formula that gives the number of rods (R) needed to build a beam of length L". Once they expressed the number of rods (R) in terms of length (L), they were asked to sketch the graph of the linear pattern and determine if the linear pattern is continuous. After the participants had produced their graphs, they were asked to explain their drawing. The linear pattern problem was chosen to give PSTs a chance to create their own understanding and solution. We attempted to infer the pre-service teachers' thinking/images from their responses while solving the task, and we probed their thinking with open-ended questions.

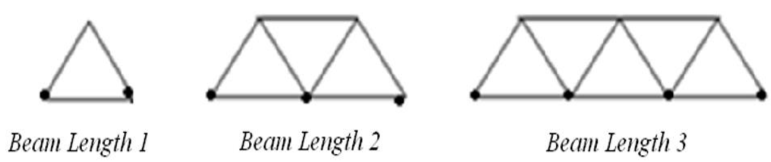


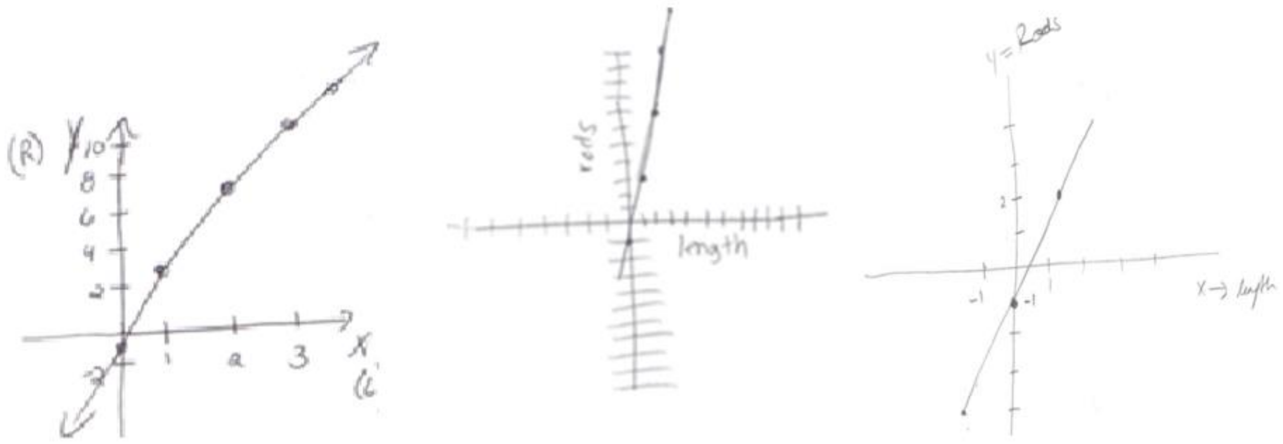
Figure 1. Algebra task

DATA ANALYSES AND RESULTS

The interviews were audio recorded. PSTs' responses and strategies were analyzed using a multiple-case study approach to describe and make inferences about their concept images and concept definitions of continuity. Analysis of the interviews revealed three common themes in students' thinking about the problem: Connecting points on the graph, ignoring the domain of the pattern, and interpreting continuity with colloquial meaning.

Tendency to Connect Points on the Graph

Among four of the PSTs, three of them (Kelly, Linda, and Marie) drew the graph of the linear pattern (R = 4L-1) by connecting the ordered points by a straight line (**Figure 2**). In **Figure 2**, the graph on the left is Linda's graph, the graph on the right is Marie's graph, and the one in the middle is Kelly's graph. This mistake may signify a shortcoming of traditional lesson about graphing functions, which often teach students to begin with plotting some points on a graph, and then connecting those points with a straight line, if the pattern is linear (Leinhardt et al., 1990). Since PSTs created the graph using the straight line, when they were determining whether the pattern was continuous or not, their decision was mostly based on their graph.



(a) Linda's graph (b) Kelly's graph (c) Marie's graph

Figure 2. Graphs with straight lines

Kelly revised the graph after she was asked to if the graph of this pattern is continuous. However, Linda and Marie did not realize the gaps on the graph and did not revise it to change a dotted line. The graph drawn by Kelly while she was solving the pattern problem was a continuous graph; however, in order to further examine her thinking the interviewer asked if the graph of the linear pattern was continuous. She responded, "I think it is definitely discrete; it should not be a line. Actually, it should be a dotted line because there is no length like 1.5 because we are looking at full triangles. Discrete." In Kelly's first graphical representation, the points were connected, but she realized that the values of *x* cannot be non-integer numbers. Therefore, she suggested a discontinuous graph which only has integer *x*-values in the domain.

Tendency to Ignore the Domain of the Pattern

Another important result from the students' responses was their interpretation of continuity without checking if the function is defined on some open interval. George was the only student who correctly graphed the pattern immediately after creating the equation (**Figure 3**). His interpretations for continuity were based on the correct graph of the pattern.

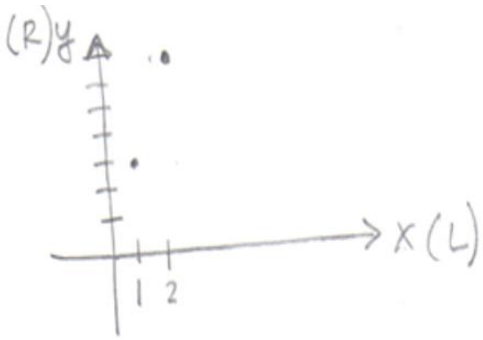


Figure 3. George's graph

> George: If I start at 0, well continuous meaning says it keeps going forever if you keep on doing this, increasingly doing this onward. Of course, you need to find the beam of length [the number of rods in the base of each figure]. The beam length would be positive. [Examines the values of L on the negative side of the x axis of his sketch] But when you get closer to negative there is no such thing as a negative beam of length so it is going to be discrete.

As seen in the excerpt, since the linear pattern had infinitely many values for positive x (or L) values and was not defined for negative x (or L) values, George said that "the graph of this linear pattern was discrete (not continuous)". Here, George did not check whether this function was defined on an open interval or not. Based on the formal definition of continuity and limit, the function must be defined on some open interval. George's reasoning for being discontinuous was based solely on "it continues on forever in both directions."

Other students' interpretation of continuity all relied on the incorrect graph of the pattern (except Kelly). Since students drew the graph by connecting points, they failed to recognize that the function was not defined on some open interval.

Tendency to Interpret Continuity with Colloquial Meaning

In addition to students' tendencies stated above, George's personal concept definition revealed another vital misinterpretation of continuity. According to George's personal concept definition continuity means "keep going forever." After his response, George was asked the following question.

> Researcher: Would the graph be continuous if it kept going in both directions, namely positive and negative?

> George: Yes, it would be continuous because it keeps going in both directions.

George's biggest concern for being continuous was incessant pattern, not the gaps between the points. The concepts of being continuous and the end behavior of going to infinity may evoke the same concept image in George's mind since he stated that "continuous meaning says it keeps going forever." The daily usage of the word "continuous" has an impact on students' mathematical interpretation of continuity (Tall & Vinner, 1981). For this reason, George might have created an incomplete concept image about continuity. He developed potential conflict factors due to his personal concept images conflicting with the graphical meaning of continuity. Stewart (2008) defined continuity as "…a function that is continuous at every number in an interval as a function whose graph has no break in it. The graph can be drawn without removing your pen from the paper" (Stewart, 2016). This potential conflict factor between his personal and formal concept definitions may hinder him from constructing complete explanations for being continuous.

DISCUSSION

This study showed some of the important aspects on PSTs' understanding of continuity. When we conducted the interviews, our purpose was to make inferences about their concept images on continuity. After conducting interviews, we identified some of their personal concept definitions and how these definitions affected their decision on whether a function was continuous or not.

Most of the students based their decisions for continuity on the graphs that they created, and only two students had a correct graph. This study can be improved by giving students the correct graph and then inquiring to make comments whether the given graph and function are continuous or not. Therefore, their thoughts on graphs' being continuous while having a discrete domain will shed more light on the study.

With this study, it has been confirmed that most of the PSTs that were interviewed did not consult the formal definition of continuity. As it was previously mentioned, most of the time students rely on their personal concept images, and until they have more experience in mathematics, they do not need to use formal definitions (Kemp & Vidakovic, 2021; Tall & Bakar, 1992). Although the pattern was a simple linear pattern, the domain of the function was discrete. Therefore, in order to determine its continuity, there was a need to consult the formal definition. In previous studies, Haciomeroglu and Schoen (2008) claimed that in advanced continuity problems, students have to consult the formal definition of continuity in order to get a correct and reasonable answer. Findings from our study indicate that students need to support their concept image with the formal concept definition, whether it is a simple or advanced task.

Our results supported the report of CUPM (2004, 2015). As it was mentioned in the report, "It is not enough for secondary mathematics teachers to have an understanding of advanced mathematics—they must also be able to connect their advanced coursework to the material they will teach." Based on the interviews conducted in this study, the preservice teachers failed to connect their advanced course work to the high school algebra task (Pattern Problem).

CONCLUSIONS

Our analyses indicated that the interviewed PSTs had an incomplete understanding of continuity. During the interviews, it was revealed that PSTs developed potential conflict factors. Although we do not know when or how these conflict factors were developed, we claim that the task we presented in this study evoked their personal images and helped us identify conflict factors in their concept images. Due to these potential conflict factors, Kelly, Linda, and Marie could not construct an accurate graphical representation of the linear pattern. Moreover, all four of them did not consider the domain of the linear pattern to decide whether it was continuous or not. George's and Marie's personal concept definition did not conform to the graphical meaning of continuity. In all of the PSTs' answers, there were some parts conflicting with the formal concept definition of continuity.

Teachers and teacher educators can generate mathematical discussions and identify conflict factors by presenting this algebra task or similar tasks that require students to do more than reproducing solutions (Oladosu, 2014). Once students' or PSTs' conflict factors are identified, their graphs can act as a common referent for the discussion. Discussions which are supported with the formal mathematical definitions may help students feel cognitive conflict. Once they experience cognitive conflict, they feel uneasy and improve their concepts as new information requires. As constructivist teacher educators, we believe that mathematical activity is a self- organization: Students develop their own questions, carry out their own experiments, make their own analogies and come to their own conclusions. Therefore, the role of the teacher and teacher educators is to establish mathematical environments which help students and PSTs to be aware of conflicts between their personal concept images and concept definitions and mathematical facts.

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