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Misconceptions and resulting errors displayed by in service teachers in the learning of linear independence

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ABSTRACT

The aim of this paper is to identify errors and misconceptions that student demonstrated when learning linear independence and linear dependence concepts. A case study is presented involving 73 in-service mathematics teachers at a university in Zimbabwe who were studying for a Bachelor of Science Education Honors Degree in mathematics. Data was generated from a content analysis of the written responses of the participants to two items from a structured activity sheet. Follow up interviews with five participants were used to gain a better understanding of their misconceptions. The study found that the participants had different kinds of misconceptions leading to errors which could be described as procedural, conceptual, and foundational respectively and the distribution of the errors differed across the two problems. For question 1 which was set within the vector space $M_{2\times 2}$, students found it harder to move past the first few two steps of formulating the general vector equation and doing scalar multiplication; those who passed those two steps were mostly able to get to a correct solution. For question 2 which was set within $\mathbb{R}^3$ most students went past the first two steps formulating the general vector equation and converting that to an augmented matrix but then made many foundational errors, most of which were related to misinterpretations of the solutions to the system of equations.

Keywords: linear independence, error, misconceptions

INTRODUCTION

Linear algebra is usually studied in first year at university level because of its applicability in many areas, such as statistics, computer science, medicine, engineering, and other aspects of mathematics (Salgado & Trigueros, 2015). Many academics have noted that the teaching of linear algebra may be challenging, and that students often struggle to grasp the ideas (Aygor & Ozdag, 2012; Kazunga & Bansilal, 2017; Possani et al., 2010; Stewart et al., 2019; Wawro, 2014). In terms of these difficulties Stewart and Thomas (2010) elaborated that students start well with aspects of first year courses by applying rules and algorithms when solving systems of linear equations and manipulating matrices but struggle to understand the concepts on vector spaces such as subspaces, span, and linear independence.

Stewart and Thomas (2019) also argued that it is the abstract nature of linear algebra that contribute to students' struggles to understand the concepts. Carlson (1993, p. 39) concurred that "…but when they get to subspaces, spanning and linear independent, my students become confused and disoriented". This showed that within linear algebra, the introduction of vector space concepts makes the work more abstract to the students and makes it harder for instructors to mediate (Mutambara & Bansilal, 2018). Edwards and Ward (2008) believed that the students' poor performance in mathematics relates to the lack of conceptual understanding. This concurs with other studies in linear algebra which claim that students developed more of procedural understanding rather than conceptual understanding of the concepts.

Linear algebra is a core subject studied in mathematics teacher education programmes in Zimbabwe and hence, there is a need to learn more about Zimbabwean undergraduate students understanding of the concepts on linear independence and linear dependence, and how they make connections between the two terms. Research suggests that as students develop their understanding of particular concepts, they develop misunderstandings and make errors and knowledge of these could be used by instructors in improving their understanding (Mutambara & Bansilal, 2021). In this study we focus on the errors and misconceptions that are revealed as they construct their understanding and make connections between these concepts. Accordingly, the following research question guided the study:

What errors and misconceptions are displayed by in service mathematics students when solving linear independence problems?

LITERATURE REVIEW

Dorier (2000) and Dorier et al. (2000) outlined that linear algebra is grounded in the theory of vector spaces, in which this study is centered on, and acknowledged that students encounter difficulties which are a result of what they called "obstacle of formalism". This term refers to the learning of concepts that require implicit reasoning and making arguments such as those required in the process of proving axioms. They further elaborated that when learning the concepts on vector spaces the students encounter new definitions as well as new theorems that they did not encounter at elementary level. They also commented that students performed badly as a result of insufficient competency with the aspects on elementary set theory, proofs, and logic in general, and manipulation of algebraic expressions. Sierpinska (2000) elaborated that if students have limited proficiency in aspects of elementary set theory, logic, and proofs, they will be labelled as being "under the spell" of the obstacle of formalism. A similar finding by Mutambara and Bansilal (2019) was that students struggle when working with vector space concepts because of non encapsulation of the prerequisite concepts of sets and binary operations. Arnawa et al. (2019) also suggested that students at the tertiary level must learn to understand definitions, the theory, and lemmas before carrying out proof related problems. Other research raised the issue that first year students have limited prior experience with proofs (Uling, 2002). Dorier (2000, p. 86) further purported that, with specific reference to concepts on vector spaces:

> Students' difficulty with the formal aspect of the theory of vectors spaces are not just a general problem with formalism, but mostly a difficulty of understanding the specific use of formalization in the theory of vector and the interpretation of the formal concepts in relation with more intuitive contexts like geometry and systems of linear equations in on which they historically emerged.

Stewart and Thomas (2010) used a combination of APOS theory by Dubinsky (2007) and the three worlds of mathematics by Tall (2004) to explain how students understand the concepts in linear algebra particularly linear independence, span, and basis. On a different note, Wawro (2014) investigated students' understanding of spanning and linear independence through the lens of concept image. In order to understand the thinking process of the students, Wawro (2014) designed mathematical activities which involved the following: describing, proving, relating, example generating, problem solving, and this provided an insight into students understanding of the concepts of spanning and linear combination. Mutambara and Bansilal (2021) also pointed out that students faced some foundational obstacles when constructing the concept of linear combination, related to inappropriate interpretations of the solutions to systems of equations.

From the above, many research papers have aimed at unearthing and finding ways of increasing learners' mathematical understanding in linear algebra. Luneta and Makonye (2010) outlined that there is the need for instructors to shift from the teacher centered methodologies and bring in the aspect of learner centered approaches where instructors view themselves as being responsible for helping learners overcome the misconceptions that they may construct. This study aims at finding out the errors and misconception that the students (who are in service teachers) make when learning linear independence concepts.

Makonye (2012) sees a misconception as a wrong belief and opinion in one's mind which results in a series of errors, while Mulungye et al. (2016) see an error as an unintentional deviation from the truth, or a mistake when solving a mathematical problem procedurally. He further elaborated that it can be caused by a misjudgment or by a student being careless. Jarrah et al. (2022) in their study of misconceptions in addition and subtraction of fractions, distinguished between procedural (involving arithmetic skills) and conceptual applications (understanding of size comparison and other problems) of fractions. In their study Hokor et al. (2022) similarly distinguished between misconceptions in probability which arose because of conceptual, procedural or interpretation difficulties respectively. Conceptual difficulties involve difficulty in understanding, and reasoning about the concept, while procedural difficulties relate to difficulties in carrying out manipulations or algorithms although they may understand the concept and an interpretation difficulty is when a student is not able to be precise and unambiguous in interpreting the concept (Hokor et al., 2022).

Agustyaningrum et al. (2018) see error analysis as an important form of assessment. He further asserted that error analysis helps the instructor identify the procedures that a student can execute, can diagnose a mathematical misunderstanding, and then select an appropriate instructional strategy to address the misconception. In line with this Makgakga (2016) also encourages the instructors to take learners' errors seriously when solving mathematical problems. The instructors must be able to identify the root cause of those errors and find ways of helping learners to correct them. Teachers should not blame learners for making errors but should embrace them as a means to improve learning (Makhubele et al., 2015).

There is little research on how the concepts of linear independence and dependence can be learned and the types of difficulties that students experience during the learning process as well as how the students reason about the relationships between two. Hence the need to find out the errors and misconception that students display when solving linear independence concepts.

In this study we use the definition of a linearly independent set of vectors by Anton and Rorres (2010), which is, as follows: Let V be a vector space over a field k. If $S = \{v_1, v_2, \dots v_r\}$ is a nonempty set of vectors in a vector space V, then the vector equation $k_1v_1 + k_2v_2 + \cdots k_nv_r = 0$ (Eq. 1) has at least one solution namely $k_1 = 0, k_2 = 0, \dots, k_r = 0$. These solutions are called the trivial solution. If this is the only solution, then S is said to be a linearly independent set. If there are other solutions in addition to the trivial solution, then S is said to be a linearly dependent.

METHODOLOGY

To explore the difficulties that students encounter when learning linear independence/dependence concepts, a qualitative design methodology was explored. Creswell (2014) points out that qualitative research is an approach that is used to address difficulties that involves a person carrying out the study to explore and understand the meaning of a phenomenon by relying on the views of the participants. This study comprised a case study of 73 students who were first year in-service teachers studying for a Bachelor of Science Education Honors Degree in Mathematics. The participants were in-service teachers as they were already practicing teachers, and did not have degrees, so could also be described as undergraduate students/ teachers. The participants were coded using tags 'T1', 'T2', and so forth, where the order did not have any significance. This was done so that the responses of the teachers could not be linked in publications to the original participants, while enabling an organization of the data.

Data for the study were generated from written responses to a structured individual activity sheet as well as individual interviews. The activity sheet contained eight questions based on the concepts of linear independence and for this article we focus on two of these items. The first item was based in the vector space $M_{2\times 2}$ consisting of 2×2 matrices and the students were asked if they are linearly independent. Item 2 was based in the vector space $\mathbb{R}^3$, where the students were presented with three vectors and asked if they were linearly independent. Five participants were interviewed about their responses to these tasks to gain more understanding about their struggles with the concepts. The participants' written and interview responses were analysed, using the analytical framework described below, which distinguished between three types of errors: conceptual, procedural, and foundation /technical errors. Some aspects of this framework have been used in Mutambara and Bansilal (2021), which focused on errors in linear combinations and spanning as well as Msomi and Bansilal (2022), which analysed errors in the area of Laplace transform.

THEORETICAL AND ANALYTICAL FRAMEWORK

This study is based on the principal of constructivism which assumes that people build up knowledge from an interaction between their experiences and new ideas (Mogashoa, 2014). Idehen (2020) argued that students come to the classroom with individual preconception which have developed in their previous experiences with related concepts and could be correct or incorrect. This means that students are considered active participants who brings into class past experiences which they connect with knew learning as they try to solve given problems. As a normal part of learning students necessarily develop misunderstandings, the knowledge of which can be used by instructors to improve their own teaching. Hence there is the need to assess students' misconceptions and errors when learning linear independence concepts and we propose a framework for analysis that distinguishes between three types of errors which can occur when students are introduced to a new concept which requires the application of previously learnt procedures or concepts.

Conceptual Errors

These errors are a result of deeply held misunderstandings where students do not have a complete understanding of the new concept (Mutambara & Bansilal, 2021), in this case, linear independence as well as not understanding the relationships between concepts in the problem. In these two problems, these were identified when the participants were not able to come up with the vector equation (Eq. 1) to test for linear independence concept, that is, say $\mathbf{0} = k_1\boldsymbol{u}_1 + k_2\boldsymbol{u}_2 + k_3\boldsymbol{u}_3$**,** or were not able to identify the method to solve the problem.

Procedural Errors

These errors arise when the student uses an inappropriate approach to solving a problem or carries out a procedure incorrectly (Mutambara & Bansilal, 2021). In this study procedural errors refer to those responses where participants were able to formulate the correct equation (Eq. 1), but failed to come up with the augmented matrix, or to calculate the determinant of the coefficient matrix.

Foundational or Technical Errors

Foundational errors are those, which occur because of poor background knowledge such as when a student incorrectly applies or misinterprets a previously learnt procedure or concept that is being applied in the new concept incorrectly (Msomi & Bansilal, 2022; Mutambara & Bansilal, 2021). According to Kiat (2005) technical errors are due to due to carelessness while carrying out a suitable procedure. In this study foundational or technical errors were those that arose in the process of solving the systems of equations, in reducing the augmented matrix or finding the determinant of the coefficient matrix. Some errors were due to slips or carelessness, while others were related to incorrect applications or misinterpretation of the solutions to the system of equations.

RESULTS

The results of the study are presented below in terms of the errors identified in the two questions followed by a summary of the errors across the questions.

Table 1. Item 1 with possible solution routes

Item 1	Possible solution routes
6. Prove that the following matrices $\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}$, and $\begin{pmatrix}0&0\\0&1\end{pmatrix}$ are linearly independent. Explain the result.	The students followed the following procedures: As a first step, consider 2×2 zero-matrix M = $\begin{pmatrix}0&0\\0&0\end{pmatrix}$. Suppose the given matrices are **A, B, C,** and **D**. Set up a vector equation $\boldsymbol{M} = k_1\,\boldsymbol{A} + k_2\boldsymbol{B} + k_3\boldsymbol{C} + k_4\,\boldsymbol{D}$. **Method 1.** Immediately recognise that the unknowns $k_1 = 0, k_2 = 0, k_3 = 0$ and $k_4 = 0$. **Method 2**: They may proceed from step 1 to set up a coefficient system of four equations with four unknowns (k_1, k_2, k_3 and k_4) or thereafter do scalar multiplication and matrix addition to set up a coefficient matrix before recognising that the unknowns (k_1, k_2, k_3 and k_4) equal 0, 0, 0, and 0, respectively. Once the solution to the system is established, then one would conclude that the set of matrices are linearly independent because of the existence of only the zero solution or trivial solution, that is $k_1 = k_2 = k_3 = k_4 = 0$.

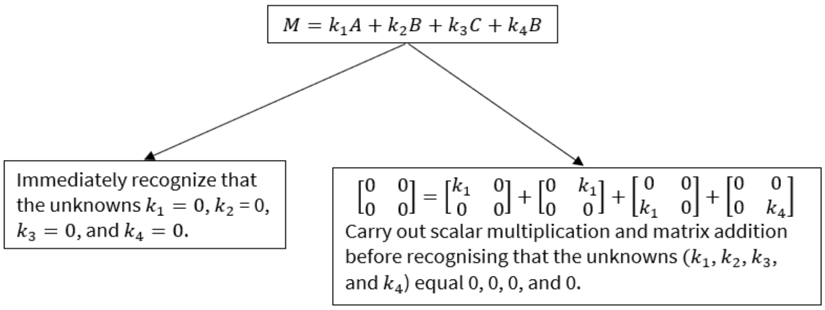

Figure 1. Summary of the methods

Results for Question 1

The question and possible ways of solving the question are presented in **Table 1**.

The methods outlined in **Table 1** can be summarized in **Figure 1**.

From the analyses, three of the students did not attempt the question while 24 were able to obtain the correct result. There were 21 responses whose errors were classified as being conceptual, 18 were considered as procedural errors and seven as technical/foundational errors. In this question we expected the students to present a sequence of steps brought together with logical connections and arguments in spelling out why the matrices are linearly independent. We now discuss the nature of errors identified in terms of conceptual, procedural, and technical.

Conceptual errors for question 1

21 of the students revealed conceptual errors. The students were supposed to come up with a vector equation of the form $k_1v_1 + k_2v_2 + k_3v_3 + k_4v_4 = 0$. The first conceptual error depicted was that by student T32 who failed to come up with the vector equation but resorted to finding the determinant of the separate $M_{2\times2}$ matrices . All the separate matrices gave a determinant equal to zero, and the student then concluded that the vectors were linearly independent. The student confused this with the idea of calculating the determinant of the coefficient matrix, not the determinant of the separate matrices thus showing a failure to appreciate the relationship involved in the problem.

The other type of conceptual error manifested by the students was that they were able to come up with the vector equation but the equation was equated to the arbitrary vector instead of the zero vector, that is $k_1v_1 + k_2v_2 + k_3v_3 + k_4v_4 = b$. This is evidenced by student T44 in **Figure 2**. 12 of the students revealed such a conceptual error. Correct step by step procedures are seen in the work of T44, as he was able to do scalar multiplication and come up with a linear combination of the vectors but unfortunately it was equated to the wrong vector. This response indicates that the student's conceptual difficulties might have originated from the confusion with the ideas of showing linear independence and spanning, because he did not have a sufficient foundation of what linear independence is. The vector w is taken as an arbitrary vector.

Six other students also demonstrated conceptual errors. These students did not come up with the vector equation. They came up with an augmented matrix with an identity matrix of order 4×4, whose solution was the zero vector. This is illustrated in **Figure 3** by student written response of student T54. These students failed to build coherent cognitive structures around the concept linear independence. In their mind it seems they were only aware that for linear independence they should get the same solution that is for the unknowns k_1, k_2, k_3 and k_4 must always be equal to zero. Another conceptual error that was similar to the above was shown by two students who also failed to construct the vector equations and simply came up with an augmented matrix with an identity matrix of order 4×4, whose solution was an arbitrary vector (a, b, c, d) as shown $\begin{bmatrix}1&0&0&0&:&a\\0&1&0&0&:&b\\0&0&1&0&:&c\\0&0&0&1&:&d\end{bmatrix}$. The solution given was $k_1 = a,\ k_2 = b, k_3 = c, k_4 = d$. This showed that these students understood the concepts as isolated facts and just memorized some rules which were not connected. This hampered the development of the required schema as mentioned by Ndlovu and Brijlall (2019) that university students only master a collection of algorithms which does not lead to a deeper understanding of the concepts.

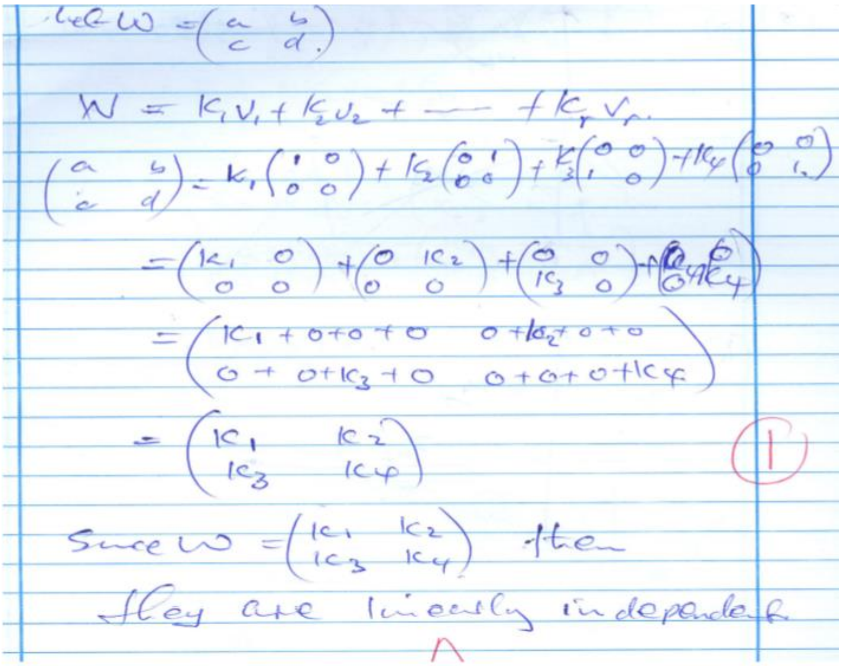

Figure 2. Written response of student T44

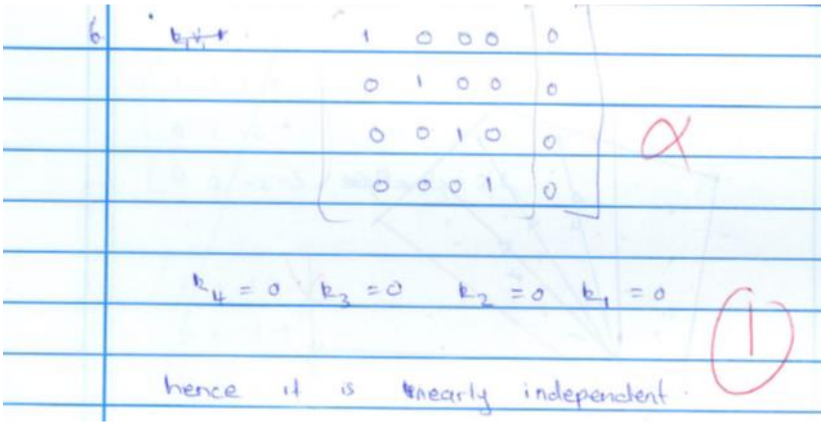

Figure 3. Written response of student T54

Procedural errors for question 1

We noted that 18 of the participants demonstrated procedural errors. One student, T30, was aware that he should come up with a vector equation equated to the zero vector. However, the equation had three scalars instead of four, but he did not even notice that at the end the solution should have four scalars, this showed a manifestation of a procedural error.

Furthermore, 17 of the participants had correct ideas and obtained the following as the final answer $k_1 = 0, k_2 = 0, k_3 = 0$ and $k_4 = 0$ without a concluding statement. However, the question required them to argue as to why the vectors were linearly independent, since it asked them to prove something, a question which needed justifications. However, no reasons were given about why the vectors were linearly independent.

Foundational or technical errors for question 1

Seven responses displayed a foundational or technical error. One student's (T40) written response is shown in **Figure 4**. The student T40 was able to come up with the correct vector equation, and step by step procedures as he is able to multiply each and every matrix by the corresponding scalar. However, the student did not complete carrying out the algebraic manipulations as he did not identify the corresponding values of a, b, c, and d, and did not provide any concluding statement. This was taken as a foundational error because there was no logical deduction or interpretation of the last step in terms of why the vectors formed a linearly independent set. This concept that the student missed of equating corresponding elements was done at school level.

Six other students' responses were categorized as a foundational error when they were able to construct the vector equations but came up with an augmented matrix as shown by student T57 written response (**Figure 5**). The student T57 did not explain how she came up with the 4×4 identity matrix despite the fact that she had the correct equation $\begin{bmatrix} k_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & k_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & k_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Student T57 formulated an augmented matrix and then solved it to get the answer $k_1 = k_2 = k_3 = k_4 = 0$. The

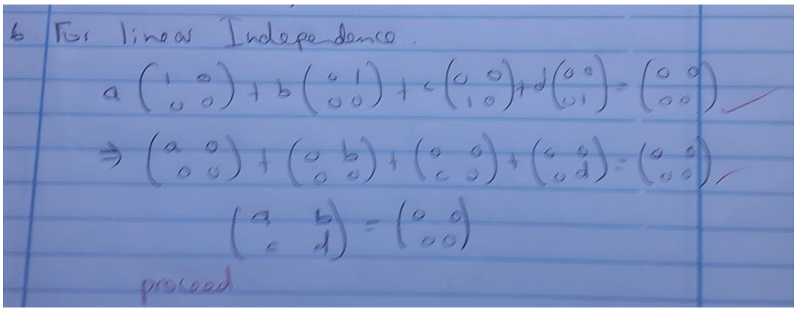

Figure 4. Written response of student T40

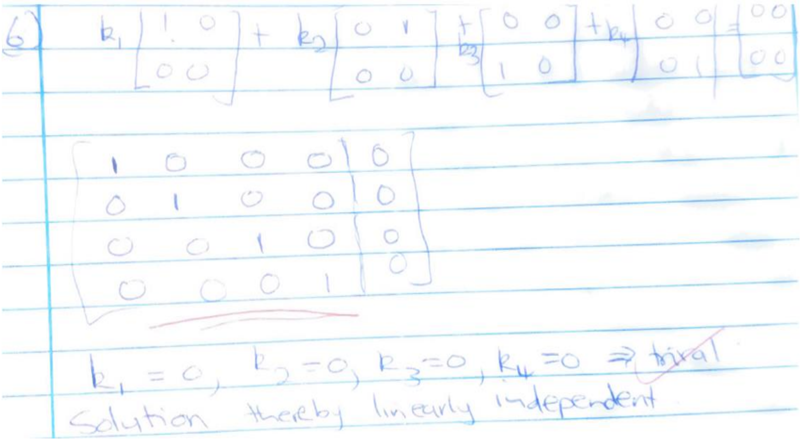

Figure 5. Written response of student T57

scalars k_1, k_2, k_3, and k_4 just appeared in the solution. It seems this participant took the sum of the 2×2 square matrix wrongly and converted it to an augmented matrix in the form $I_4x = 0$. Thomas and Stewart (2011) asserted that students are reluctant in using other forms of intellectual reasoning but are just confident with matrix procedures. On these items, students were expected to go beyond just manipulating algorithms, and illustrate their background knowledge about matrices and their properties. The student here could have added the corresponding elements, and then finally equated corresponding elements. Addition of matrices is a concept that is learnt at elementary level and therefore this error is seen a technical/foundational error. This inadequate conception showed that the students struggled to show that given vectors in matrix form are linearly independent, and such errors have not been identified in many studies about linear independence.

Interview responses related to question 1

An interview was conducted with student T57 whose response is shown in the following excerpt:

> **R**: In order to show that it is linearly independent, you were able to write the equation $k_1v_1 + k_2v_2 + \cdots k_nv_n = 0$. How then did you proceed from there?
>
> **T57:** I took the components in the first row first column. This one was matching it with this one. [attempt to come up with the 4×4 matrices augmented to the zero matrix]. I think thus the same confusion with the other ones we have discussed earlier.
>
> **R:** So, what did you do next?
>
> **T57:** Like what I said, I was taking the first element in the first row first column, then I write zeros, I take again the next element in the next row, this one is the second-row number first row second column, was this one, and I continued. Then I used back substitution to obtain the values of k_1, k_2, k_3, and k_4. Since the values of k_1, k_2, k_3, and k_4 were the same then it is linearly independent.

The discussion with T57 about her formulation of the 4×4 square matrix is an indication of a possible misconception which she continued to hold on to. The student applied inappropriate rules in an attempt to simplify the vector equations and demonstrated confusion in an attempt to show that the given matrices are linearly independent. However, the student had some idea that at the end she should have trivial solutions so that she could assert linearly independence.

Another discussion with T44 concerned his struggles to outline whether the vector equation was supposed to be equated to an arbitrary vector or to the zero vector. After much probing, he was able to state that he was supposed to equate the vector equation to the zero vector. The interview helped him to identify his misconception. The interview response supports Metcalfe (2017) contention that whenever people make an error, it is helpful to give them feedback and ion out where they went wrong.

7. Determine whether or not the vectors $u = (1, 1, 2)$, $v = (2, 3, 1)$ and $w = (4, 5, 5)$ in $\mathbb{R}^3$ are linearly independent.

Figure 6. Item 2

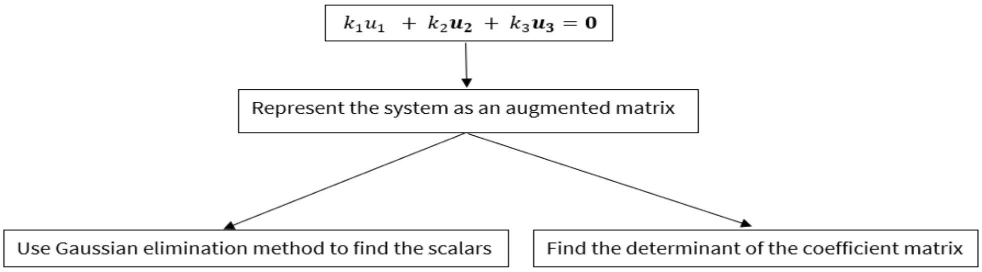

Figure 7. Flow diagram of the two methods

Results for Question 2

Out of the 73 students, two did not answer the question, 26 of the students got a correct result, seven revealed conceptual errors, two procedural errors and 36 revealed technical/foundational errors. Item 2 is presented in **Figure 6**.

To solve this problem the students could work through the following steps:

1. Set up a vector equation $\boldsymbol{0} = k_1u_1 + k_2\boldsymbol{u_2} + k_3\boldsymbol{u_3},$ where k_1, k_2 and k_3 are scalars.
2. Set up a system of three homogenous equations with three unknowns (k_1, k_2, and k_3).
3. Represent the system as an augmented matrix.
4. Carry out row reductions on the matrix and interpret the reduced matrix as indicating that that the third column has no pivot.
5. Can also find the determinant of the coefficient matrix.

As explained above, the students could have opted for two different methods, which are expressed in **Figure 7**.

In order to use Gaussian elimination method, only three elementary row operations were required to reach the correct conclusions as shown below $\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{4}\\ \mathbf{1} & \mathbf{3} & \mathbf{5}\\ \mathbf{2} & \mathbf{1} & \mathbf{5}\end{bmatrix} \sim \begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{4}\\ \mathbf{0} & \mathbf{1} & \mathbf{1}\\ \mathbf{0} & \mathbf{-3} & \mathbf{-3}\end{bmatrix} \sim \begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{4}\\ \mathbf{0} & \mathbf{1} & \mathbf{1}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}$ and the following interpretation was necessary: the third row does not have a pivot. Thus, the vectors are linearly dependent. Students here revealed conceptual, procedural as well as technical/foundational errors.

Conceptual errors for question 2

Two of the students did not attempt the question and five of them applied inappropriate rules so provided incorrect responses indicating some conceptual errors. The students P44 and P46 formulated vector equations of the form $w = k_1v + k_2u$. They substituted the vectors and came up with three equations in two unknowns. They solved the first two equations and obtained the values of $k_1 = 2$ and $k_2 = 2$. The values of k_1 and k_2 were substituted into the third equation and obtained the statement $5 \neq 4$, and made some conclusions. The students revealed a conceptual error because they used the method for determining whether a vector was a linear combination of a set of given vectors, without going further.

Procedural errors for question 2

There were two responses which displayed procedural errors. Two of the students T13 and T33 were able to come up with a vector equation in three unknowns and equated it to the zero vector. Student T33 was able to carry out step by step procedures as was able to do scalar multiplication and come up with a system of linear equations. The student was able to come up with an augmented matrix but was stuck as he did not carry out any elementary row operations and did not try to calculate the determinant. The written response of the other student T13 is shown in **Figure 8**.

It is evident that student T13 was able to come up with the vector equation but struggled to come up with the system of equations in three unknowns showing some knowledge gaps in the construction of the concept of linear independence. However, she made up an augmented matrix, but it seemed that she could not figure out what the next step should be.

Foundational or technical errors for question 2

This was the most common error revealed by 36 students. 15 students used the Gaussian elimination method but encountered a number of technical or foundational errors such as problems with basic manipulations of figures. There were many computational errors with directed numbers. An example is that of participant T8 who carried out some row operations and obtained the following matrix $\begin{bmatrix}1 & 2 & 4 & 0\\ 0 & 1 & 1 : & 0\\ 0 & 0 & -3 & 0\end{bmatrix}$ after making a few computational errors. After getting the reduced matrix, she further revealed another foundational error by writing that $-3 \neq 0$ and then concluding that there is inconsistency as the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ cannot be expressed as a linear combination which means there is linear independence. This shows that there are mix up of ideas in trying to interpret solutions to system of equations.

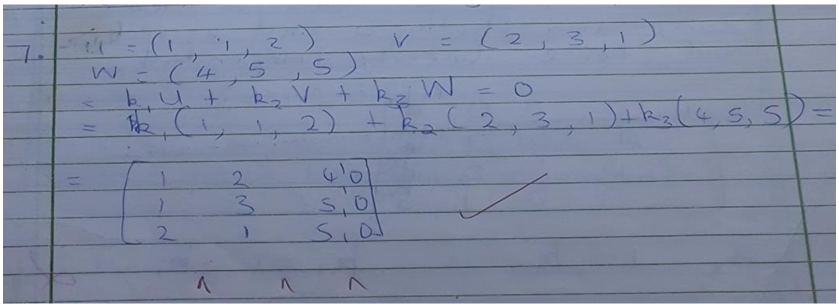

Figure 8. Written response of student T13

Some of the students carried out many algebraic manipulations during their row reduction attempts, obtained incorrect solutions and were not able to write any conclusions. 12 students were able to carry out the correct row operations but did not provide the correct deduction. Some of the participants, like T29, T38, T52, and T55, made the following conclusions respectively: the system is inconsistent therefore it is linearly independent; the vectors cannot be expressed as a linear combination because it has many solutions. This showed a mix up of ideas and revealed weaknesses in the students' understanding of the concepts. They could not abstractly outline the conditions necessary to reach the correct arguments for linearly independent. They only used the terms interchangeably. Dubinsky (2007) asserted that if the prerequisite concepts and not mastered, then the understanding of the new material is impeded.

Some of the students realized that they could use the determinant method, for example, student P40. However the student made an error in transcribing the vectors and at the end was not able to get the correct determinant value of zero. This demonstrated an error which Siyepu (2013) referred to as a slip, which illustrates a technical error. Another student, P60 obtained an incorrect determinant value after making mistakes in basic algebra manipulations. Some students failed to use the determinant method correctly, for example student T7 who used the Laplace expansion method incorrectly and obtained the wrong determinant which is also a foundational error. There were also 6 instances where the participants calculated the determinant without treating the determinant as a function thus displaying foundational error. This was because of students not grasping the concepts related to determinants which was covered in their first Linear Algebra module. They failed to recognize the difference in meaning between the following brackets () and | |. These results coincide with the findings in the literature by Kazunga and Bansilal's (2017) study.

Interview responses related to question 2

The following exchanges took place with student T23.

R: How do we show that the following vectors are linearly independent? (Showing the student question).

T23: [quiet for a while] …. hmm this one pointing to the vectors, [the student nodded so that he can proceed], I will come up with a matrix.

R: What do you do with the matrix?

T23: I will reduce it to row echelon form.

R: Can you explain the end result so that you can tell that the set of vectors is linearly independent.

T23: I must get a row of zero at the end.

R: So, what is your conclusion?

T23: Hmm. I should get what do we call these, you must get hmm the solutions.

R: What types of solutions?

T23: Trivial solutions

R: What are trivial solutions?

T23: [laughing for a moment]. Hmmm. The questions are too many now. I do not know how to explain this now but what I am saying now is that we should not be found to be using parameters at the end of the solutions.

Student T69's response above indicated that he was not able to give a precise description of the procedures that are used to determine whether given vectors are linearly independent or not. Though the student attempted to react from a series of instructions, his explanations demonstrated an incomplete conception of linear independence concept. The student struggled to accommodate the new learnt material.

Table 2. Summary of conceptual, procedural, and foundational errors

	Not attempted	Conceptual errors	Procedural errors	Foundational and technical errors	Completely correct
Question 1	3	21	18	7	24
Question 2	2	7	2	36	26

Student T63 and T25 had this to say:

T63: This is a 3×3 matrix right therefore I can find the determinant.

R: The determinant of what?

T63: The determinant of the vectors. If the determinant is zero, then the vectors are linearly dependent.

R: Which other method can we use to check for linearly independence?

T63: The determinant method only.

R: Let's say you are given a hypothetical matrix reduced to row echelon form as shown below what would be your conclusion in terms of linear independence.

$$\begin{bmatrix} 1 & 1 & 2 : 0 \\ 0 & 1 & 2 : 0 \\ 0 & 0 & 4 : 0 \end{bmatrix}$$

T63: Will be having solutions that are different. This will mean the matrices are linearly dependent.

T25: [writing it down and started to do back substitution] The solution are unique solutions, we get x_3=0… all of them are zeros meaning that the set of vectors is consistent hence the vectors are linearly independent.

R: Which other method can you use to determine linear independence?

T25. The determinant method, that if the $det = 0$ the vectors are linearly dependent and if the determinant is not equal to zero, then it is linearly dependent.

From the dialogue with T63 we observed that he attempted to describe in words the procedures to be followed when determining linear independence. However, T63 was not very fluent in the discussions and missed some points showing that he did not have control over the transformations that he was carrying. Furthermore, we observed that T63 failed to construct the concept of the solution of systems of equations, and he could not connect the existence of trivial solutions to the concept of linearly independent vectors.

From the hypothetical question, student T25's explanations were convincing, precise and gave an argument that is mathematically convincing. Her written response indicated that she used the determinant method correctly and was also able to view the determinant as function. She obtained the correct determinant with correct interpretation. NCTM (2000) sees reasoning as a vital attribute in mathematics and asserted that it helps students to come up with correct logical arguments.

Summary of Errors

An overview of the different types of errors for the two questions appears in **Table 2**.

DISCUSSION

Discussion of Conceptual Errors for Linear Independence

In question 1, many students could formulate the vector equations, but they equated it to an arbitrary vector instead of the zero vector of the $M_{2\times 2}$ vector space. 12 students manifested such an error which is conceptual in nature. Some of the students were confused and they constructed a 4×4 square matrix and augmented it to either a $M_{4\times 1}$ zero matrix or an arbitrary one, as seen in **Figure 3** with the written response by student T54. The students who equated it to an arbitrary vector supported the contention by De Lima and Tall (2008) that they were building on the experiences that they had learnt before on the aspect on spanning, instead of equating it to the zero vector. The specific concept image has been used out of its domain of validity, and according to Makonye (2012) the existence of such contradictions in their concept image hinders the students in constructing new mathematical knowledge. However, after formulating the incorrect vector equations, some students were able to carry out correct procedures as they were now able to carry out scalar multiplication and vector action, but at the end obtained the results $k_1 = a,\ k_2 = b, k_3 = c$ and $k_4 = d$. Kazunga and Bansilal (2017) argued that students should have both procedural and conceptual knowledge since both are basis for the learning of linear algebra concepts. Here we can see that the students were able to execute the correct procedures, but because of their conceptual errors, the results they obtained were incorrect.

Discussion of Procedural Errors for Linear Independence

There were 18 and two students who displayed procedural errors in question 1 and question 2, respectively. The total number of procedural errors was small, and this concurs with Ndlovu and Brijlall (2016) study that the undergraduate student learning is more connected with procedural understanding rather than conceptual understanding. This is in line with Malambo (2021) who also noted that students mastered procedural knowledge without understanding the conceptual connections in relationships. It is therefore important that instructors should ensure that discussions related to conceptual connections are not ignored in class.

Discussion of Technical and Foundational Errors for Linear Independence Questions

In question 1, the number of algebraic manipulations was limited such that only seven of the students revealed technical or foundational errors. The students could quickly figure out the solution after carrying out a few steps. However, for question 2, there were 36 participants who displayed some technical or foundational errors when they attempted to find out whether the three vectors were linearly independent or not. The question required the use of Gauss elimination method or the method of calculating the determinant. These methods involve a series of step-by-step procedures. An analysis of the students' responses showed that they made many technical errors while trying to perform the necessary algebraic manipulations. To add onto that, it was evident that the students had some more challenges as they failed to interpret the solution to the system of equations in terms of the concept of linearly dependent/independent vectors. For example, considering the answer obtained by student T8, as shown, $\begin{bmatrix} 1 & 2 & 4 & & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & -3 & & 0 \end{bmatrix}$ the student was supposed to obtain the values of $k_1 = k_2 = k_3 = 0$. Instead, student T8 wrote that $-3 \neq 0$ therefore there is inconsistency hence the vectors cannot be expressed as linear combination which means there is linear independence. All of these are considered as foundational errors which arose because of a limited understanding of the concept of solutions to a system of equations. Fifteen students revealed calculation or technical errors. The work by Cobb (1994) and the one by Sfard (1991) imply that the weak prior knowledge relating to the solution to systems of equations has acted as a barrier to the development of the abstract concept of what linear independence is. Another group of 14 students also demonstrated foundational errors due to a failure to link a matrix with a zero-vector obtained with the concept of linear dependence. This is a result of a failure to interpret the solution of systems of equations and their relationship to linear independence/dependence. Celik (2015) argued that to facilitate learning, there is the need to analyse epistemologically these concepts that students fail to understand. The other drawback that led to the development of technical/foundational errors was failure to apply Sarrus rule or Laplace method of expansion when calculating the determinant. This again brings to mind that the students' concept images might have some contradictory ideas which result in them not recognizing the appropriate techniques.

Differences in Error Patterns Across the Two Questions

The results show that for question 1 which was set within the vector space $M_{2\times2}$, students found it harder to move past the first few two steps of formulating the general vector equation and converting that to an augmented matrix representing a system of equations corresponding to the components of the vectors. For those students who managed to move past those two steps, they were mostly able to get to a correct solution. For question 2 which was set within $\mathbb{R}^3$ the situation was slightly different. Most students went past the first two steps but faced obstacles in terms of foundational and technical errors, most of which were related to misinterpretations of the results obtained from the solution to the system of equations, or to technical errors arising from incorrect algebraic manipulations. Many were unable to coordinate the results that they obtained with the original vector equation (Eq. 1) that they started, that is, they were not able to make valid conclusions in the light of the kind of solution they arrived at. In a previous study, Mutambara and Bansilal (2021) found similarly that the students' poor background in systems of equations led to most of the errors they made in working with linear combination tasks.

CONCLUSION

Overall, this study has shown that the in-service mathematics teachers displayed many errors and misconceptions when solving problems based on linear independence concepts. These students displayed some conceptual, procedural, and technical or foundational errors in their responses to items based on linear independence. The written responses provided an insight into the nature of the conceptual, procedural, and technical errors. The interviews also added an in depth understanding into what the students think and helped identify whether their misconceptions are persistent. However, it is evident that some of these misconceptions were more serious than others. Many participants made multiple errors on a single question. Metcaffe (2017) is of the idea that immediate feedback must be given because it helps by giving opportunities to explore and analyse the fundamental problem thus leading to the correct answers. Most of the errors across the items were technical or foundational in nature followed by conceptual errors. This showed that most of the students struggled with the prerequisite concepts, and they could not make interrelationships with the given concepts. Some of the students did not even know where to start. We noted that 43 revealed technical or foundational errors and 28 conceptual errors and 20 procedural errors. Many of the technical or foundational errors were because of misinterpretations of solutions to systems of linear equations. One common area where students displayed persistent misconceptions was that of interpreting the results after carrying out Gaussian elimination (the method of solving systems of equations). Students could not make a connection between the row reduced matrix to the concept or definition of linear independence.

Celik (2015) argued that the instructors should provide appropriate pedagogical support so as to enable the conceptualisation process. This will help to rectify students' misconceptions. Knowledge and understanding of these errors are important because they can provide guidance to the instructor in structuring the learning materials so that it can take these unexpected errors into account and help students improve their learning (Makgakga, 2016; Metchafe et al., 2017). Based on this recommendation, it is important that learners must be given opportunities that allow them to experiment with different types of system of equations, carrying row reductions, and interpreting systems of linear equations, adding matrices before engaging with the concepts of linear independence. We also noted that the class taught was very large. This might have prevented them from engaging constructively with the concepts. It is advisable to have smaller groups of the students during tutorials so that they get focused attention from the instructor as they struggle with these concepts. This can be seen as a major goal that will help students to reach the structural level of understanding.

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