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Formation and development of mathematical concepts: Elements for research and teaching

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ABSTRACT

This article presents a theoretical-didactic perspective on the formation and development of concepts concerning mathematical objects as a result of research on didactic difficulties in dealing with concepts in school. The theoretical foundation that supports the research is based on the contributions of the materialist approach to the theory of knowledge. Specifically, an interpretation of its application in the teaching of mathematics is under investigation, specifically in the process of formation and development of concepts in the field of Euclidean geometry. However, the elements that are derived are essential for the treatment of concepts in any sub-branch of mathematics. The analysis of this process and the factors that influence the development show some fundamental elements to consider in the research activity in the field of mathematics education.

Keywords: mathematical object, concept, generalization

INTRODUCTION

From a theoretical-didactic perspective, important research (Arteaga et al., 2009; Hernández-Gómez et al., 2019; Ramos & López, 2015; Winicki-Landman, 2006) suggests that mathematical concepts are a special category that should be considered in mathematics teaching, as they constitute the fundamental basis with which mathematical thinking operates. Through their formation and development, the objective of establishing and representing the relation between mathematics and objects whose existence is independent of human consciousness (objective reality) is favored (Ballester, 1992). These works agree that the study of mathematical concepts and their treatment in the school environment can lead to the following achievements:

1. Understanding concepts and their definitions is fundamental to understanding mathematical relation.
2. It provides the basis for applying what has been learned in a justified and creative manner.
3. The formation and definition of concepts represent an essential point for logical-verbal training.
4. The possibility of transmitting important ideological notions in the elaboration of concepts and their definitions, which allow for the fixation of qualities in the personality of the learner.

In this regard, research studies reported by Angulo et al. (2020), Hernández-Gómez et al. (2019), and Locia et al. (2018) have highlighted the importance of treating mathematical concepts as an essential part of the teaching and learning of mathematics. They generally agree that many of the problems related to understanding mathematics are rooted in the insufficient treatment of mathematical concepts and the lack of methodological preparation for their incorporation and development as teaching objects in the classroom.

In the literature, research has also identified that both students and teachers at different educational levels have significant difficulties with mathematical concepts, which hinder their understanding and development of various mathematical contents. Among the essential difficulties are those related to the definition, identification, and application of concepts (Morales & Damián, 2021a, 2021b; Morales-Carballo et al., 2022). These difficulties may be due to the lack of conditions for understanding concepts and their definitions in learning activities (Hernández-Gómez et al., 2019), teacher training, curriculum approaches, among others.

Based on the referenced research, we identify the relevance of studying mathematical concepts and their assimilation as a means to develop mathematical knowledge. However, this position is still in development and often differs from what happens in practical activity at different educational levels. It is commonly observed that in these practices, definitions are repeated to students as they are stated in texts (Hernández-Gómez et al., 2019). This practice is undoubtedly not sufficient for developing students' comprehension of concepts and mathematics in general. For example, in the study plans and programs, at least in Mexico, the content on mathematical concepts proposed for treatment in school is generally presented as finished knowledge and not as a process. This means that the activity on concept reconstruction and definition processes is not promoted. The authors argue that this absence does not allow students in training and even the teacher to reflect on these processes of elaboration and development, which are essential, since they influence understanding.

The aim of this work is to contribute important elements to research in mathematics education on the formation and assimilation of concepts and to provide insights into the highlighted issues. The objective is to develop and exemplify a theoretical-didactic perspective for the formation and development of mathematical concepts, and to demonstrate its application in the field of Euclidean geometry.

THEORETICAL & METHODOLOGICAL FOUNDATION

In the literature, different theoretical models have been identified that, from different perspectives, have been oriented towards the study of understanding in mathematics, seeking to describe and propose alternatives to influence this problem. In this direction, Sierpinska (1987) considers the possibility of understanding as the overcoming of cognitive obstacles, i.e., from this viewpoint it is assumed that overcoming an obstacle means that the student or the one who constructs his knowledge must overcome his convictions and analyze the beliefs about the situation (the obstacle) from an external viewpoint. In this way, it will be possible to prepare the conditions for identifying the objects associated with the concept, the invariant properties and the step to the generalization of the scope of application of the concepts, then synthesize the relationships between properties, facts and objects that are fundamental processes in such orientation for overcoming an obstacle. Another theoretical model of understanding is the one presented by Vinner (1991), from this theory the understanding of concepts is acquired when the student constructs an image of the concept (the collection of mental images, representations and related properties attributed to a concept) and as the image of the concept is developed, apparently conflicting images can be identified in this process, overcoming these situations leads the process towards the formal definition. The construction of concept images occurs when new information is incorporated and faces the consolidation of this information within the cognitive structure already present in the student. In this sense, Tall (1991) considers that the notions of assimilation and accommodation influence the incorporation of information in the process of concept image development, thus favoring the stages of generalization and abstraction.

As can be identified in previous lines, the theoretical models of reference do not contradict each other; through different paths they address common aspects that influence mathematical understanding. In the present research it is argued that the contribution of the elements for the understanding provided by the theoretical models of reference are included and enriched in the theoretical-didactic perspective for the formation and development of mathematical concepts presented in this work, which is based on the principle of apprehending is knowing, in the theory of knowledge, theory of activity and is nourished by the authors' own vision.

Theory of Knowledge

Based on the principle that apprehend[1] is knowing, it is logical to think that anyone who wants to apprehend mathematics must be able to carry out processes of constructing mathematical knowledge, and therefore, guidance is needed on how these processes occur. The materialist approach of the theory of knowledge (Guétmanova, 1989) establishes that the cognitive process follows a schema consisting of six categories: sensation, perception, notion, concept, judgment, and reasoning. The first three categories correspond to what is called sensory or material knowledge, that is, the different forms of thought originating from direct reflections of concrete reality–material–on the human consciousness. The last three categories correspond to what is called abstract thought, which are the different forms of thought originating from indirect reflections of subjective reality–non-material–on the human consciousness. The researcher affirms that abstract thought, through its three essential forms: concept, judgment, and reasoning, is the means for constructing theoretical knowledge.

> We know the laws of the world, the essence of objects and phenomena, and their commonalities through abstract thinking, the most complex form of knowledge. Abstract or rational thinking reflects the world and its processes in a more complete and profound way than sensory knowledge. The transition from sensory knowledge to abstract thinking is a leap in the cognitive process, a leap from knowledge of facts to knowledge of laws (Guétmanova, 1989, p. 13).

According to this approach, knowledge starts with live observation, that is, the set of sensations and perceptions through which the environment is reflected in the human brain (Campistrous & Rizo, 2003). However, the type of knowledge generated at this stage is always of a material or empirical nature. On the perceptions, processes of analysis, synthesis, abstraction, and generalization are carried out to give rise to the formation of concepts, judgments, evaluations, relations, among other important components of the content to be learned. In this way, a deeper stage of knowledge is reached, one that manages to abstract the most internal or essential features of objects and the invariant relation between them; the most stable regularities in phenomena, and the most refined algorithms in processes. This higher knowledge can be achieved through the action of abstract thinking in its three categories: concept, judgment, and reasoning, which make up the second stage in the elaboration of knowledge. Once this knowledge has been developed, it is applied in practical activity, where the need to develop such knowledge was found.

[1] To apprehend commonly implies a material retention, but in this document it is used to signify a retention in the abstract realm. Thus, in the process of knowledge construction, *conceptual apprehension* involves retaining the meaning–through the formation of a mental image–of the essential characteristics of the object of knowledge.

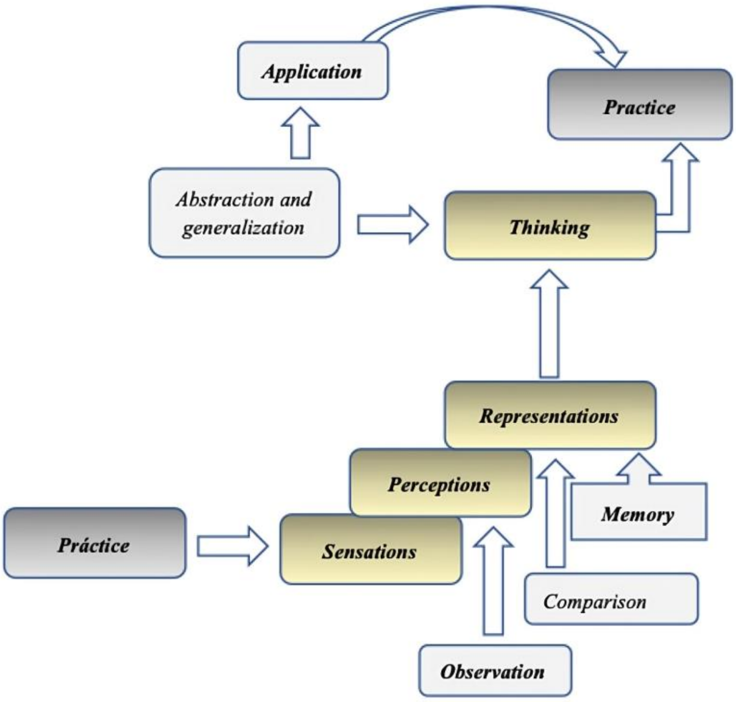

Figure 1. Representation of process of knowledge elaboration (Rizo & Campistrous, 2003)

In this regard, Rizo and Campistrous (2011) reinforce the justification that the path of knowledge begins with live perception (in practice) and culminates in practice, in qualitatively superior conditions after being enriched by a process of intellectual elaboration. **Figure 1** shows an approach to the processes of thought that favor this dialectical path.

Specifically speaking, the concept is the knowledge built on objects, phenomena or processes of reality; the judgment is a logical proposition whose content is concepts or relation between concepts, that is to say, its content is knowledge; finally, the reasoning or argument comes in different forms: ponendo ponens, tollendo tollens, among others, which are usually identified as a logical structure in which from a set of propositions considered as true–judgments–it is possible to obtain as a conclusion, a new proposition, that is to say, new knowledge.

The theory of activity (Leontiev, 1981) establishes that, in practical activity, people interact with the objects of reality and from them they apprehend their meanings; in other words, they construct knowledge about them. Such activity–cognitive process–and in general any other, requires a set of actions and operations to be carried out. According to this approach and that of the theory of knowledge, the process of knowing about the objects of reality can be identified with the process of concept formation (PCF)[2]. Rubinstein (1969) has identified the internal–theoretical–actions and operations of thought (IAOT) that carry out this process; these are analysis, abstraction, generalization and synthesis, and for their realization they can be supported by practical actions and operations: Analysis: consists in the breakdown of the object, phenomenon or process, in the clarification of its elements, features, parts, data or aspects, and in the determination of the dependencies or relations that may or may not be essential. By means of analysis we can differentiate the essential features, relations or operations from the incidental or inconsequential non-essential ones. Abstraction: it consists of dispensing with the non-essential features but highlighting the essential and common ones to the objects studied. It is the mental process by which one passes from sensible qualities to abstract qualities. Generalization: it originates mainly at the level of action. It consists in verifying whether the set of essential and common features abstracted can be extended to all the objects of the collection. In this operation, the extension of the concept is delimited, i.e., the class to which the objects that meet the generalized features can belong. Synthesis: it is a theoretical mental action by which the object is reconstructed through the establishment of its essential characteristics. But this mental action is supported for its realization, in an external action, which consists in "crystallizing" or "depositing"3 the meaning of the generalization produced, on a sensitive and concrete support–some representation–. That is to say, it can be represented in a definition, a proposition or a procedure, according to the type of abstracted and generalized features.

Process of Concept Formation

PCF is a cognitive process that takes place through IAOT: analysis, abstraction, generalization and synthesis (Majmutov, 1983).

From the teaching point of view, we will describe this process of knowing about objects, phenomena or processes of reality, making particular reference to PCF on a mathematical object (MO)[4] with the support of **Figure 2**, emphasizing that the elements identified in the representation are essential within the same theory of knowledge.

[2] PCF means from now on, process of concept(s) formation.
[3] The use of the terms "crystallize" or "deposit" could be said to be out of context, however, they are used with the purpose of simplifying the discourse. Therefore, they will be used hereinafter even without quotation marks.
[4] MO means from now on, mathematical object(s).

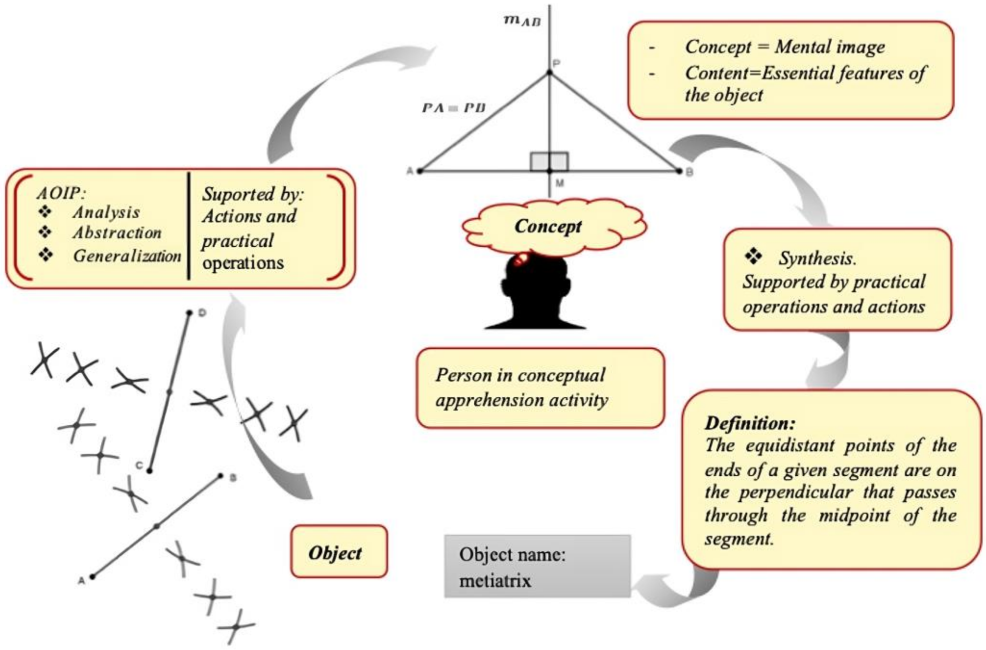

Figure 2. Concept formation process (Source: Authors' own elaboration)

This process starts with the analysis of the representation of a situation outside the studied object; in fact, we have not built MO yet, so there can be no representation of it. With the support of internal or external actions and operations, the analysis is performed to differentiate the features of the studied situation, in order to distinguish the essential ones from the non-essential ones. By means of abstraction, we can dispense with the non-essential features and at the same time highlight the essential ones; this internal action can be supported by practical operations such as highlighting or coloring the essential features on the representation, or separating from the representation the features we wish to dispense with. If, when reviewing other similar situations, the previous features are presented, then it is possible to generalize them as essential and common features of a class of new objects.

As a result of the realization of these internal operations of thought, an image of a representative of this class of objects is formed in our mind; it is a mental image–concept about MO–formed on the basis of the essential and common features of the objects it represents. In the final part of PCF, synthesis is the internal action of thought that reconstructs MO on the basis of its common and essential features; for its realization it relies on an external action, which consists in crystallizing or depositing the meaning of the produced generalization–knowledge built on MO–on a material representation, according to the type of abstracted and generalized features. But the features enlisted in the representation must be the same ones on which our mental image or concept about the constructed MO was formed.

Regarding this cognitive process, it is necessary to say that some authors, among them (Davýdov, 1982), designate in a summarized form the whole PCF with the term generalization. Thus, this term in this work will have two meanings: the first as one of the four IAOT that carry out PCF; and the second, as the entire cognitive process. 1. According to their form, generalizations can be recognized under different names: formulation of procedures, recognition of regularities, formulation of definitions, formulation of results, formulation of propositions, recognition of patterns, determination of invariant relations, among others. 2. According to their category, generalizations can be theoretical or empirical. When the teacher has been guiding and accompanying PCF on an MO, a specific way to evaluate the category of the generalizations produced by the students, it is necessary to keep in mind: If IAOT realizers of this cognitive process are supported by theoretical operations; or are produced on the basis of internal, essential and common features, it is a theoretical generalization. When IAOTs realizing PCF are supported by practical actions and operations; or are produced on the basis of external or non-essential features, it is an empirical generalization. When the teacher did not guide or accompany PCF on an MO, he/she cannot know under what conditions the generalization was formed or produced; therefore, a specific way to evaluate the category of generalizations consists in reviewing the arguments used to assert them: theoretical, if the arguments used to support them are based on the essential internal features, that is, on a theoretical basis. Empirical, when the arguments offered to assert them are based on non-essential external features, or on beliefs, or when an object does not fall within the generalization volume of the concept, and yet is assumed to have the property of the objects within the generalization volume, i.e., on an empirical basis.

Depending on the type of characteristic of MO under study, generalizations–concepts–can be classified into three types: concepts about objects, about operations, or about relations (Ballester, 1992; Jungk, 1985). Concepts about objects designate classes of real or ideal objects, which can be characterized by means of representatives. Examples: Triangle, quadrilateral, angle, number, etc. Concepts about operations designate actions that are performed with objects. Examples: bisection of an angle, addition, multiplication, etc. Concepts about relations reflect the existing relation between objects. Examples: perpendicularity, parallelism, congruence, Pythagorean relation, proportionality, among others.

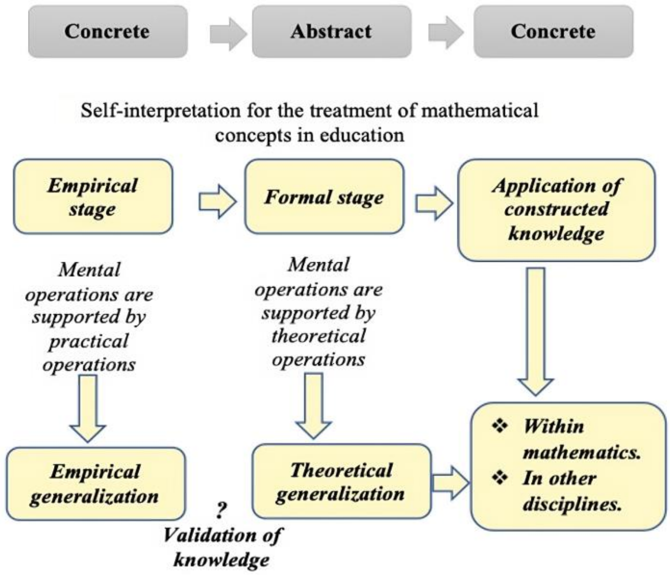

Figure 3. Stages of knowledge in construction process according to materialist view of theory of knowledge (Source: Authors' own elaboration)

Based on the principle assumed here–*to apprehend is to know*–, on the orientations from the materialistic point of view of the theory of knowledge, on the focus on the theory of activity and the own vision of the authors of this work, a didactic perspective for teaching, in particular of the concepts on MOs (DPTCMO)[5] is formulated; whose structure considers the three stages in the process of construction of mathematical knowledge, according to the theoretical elements that sustain it (**Figure 3**).

Thus, the process of construction of mathematical knowledge from such a perspective begins in the concrete with an empirical stage in which mental or theoretical operations are supported by practical operations–measuring, moving, superimposing, comparing, among others–that can be performed on representations in general, producing empirical generalizations. The passage from the concrete to the abstract will occur at the moment when the same generalization is obtained through the realization of mental or theoretical operations supported on theoretical principles–as a logical consequence derived from mathematical principles–. This second–formal–stage is the mathematical demonstration or deduction, and its function in this process consists in assuring as valid the knowledge produced in the first stage or transforming its category from empirical to theoretical. The third stage of application of knowledge corresponds to the passage from the abstract to the concrete; it will occur at the moment when a field of application of the validated knowledge is found in practical activity. Of course, knowledge is also constructed in the step called from the abstract to the abstract–applications of knowledge within mathematics–, in this case, IAOT, supported by previous results, can generate and validate new knowledge, which can then go from the abstract to the concrete.

Methodology of mathematics teaching (Ballester, 1992; Hernández-Gómez et al., 2019; Jungk, 1985) reflects in part this perspective, when it points out that with respect to the treatment of concepts about MO, two ways can be distinguished in general: deductive, whoever goes through this way, his process of knowledge construction, begins with the presentation of the generalization that describes the object of knowledge, that is, the representation–in some of its forms–of MO of which he wants to know. In this process, internal or theoretical actions and operations are supported by external or practical actions and operations; the latter are performed on such representation, through the use of examples–and counter-examples–, seeking to apprehend the meaning of the knowledge about MO, deposited in the representation, that is, to realize the common and essential features of the object, and it is expected that, upon achieving such meaning, the same generalization from which it started, can be established.

Inductive way, by means of this process the activity of generalizing is carried out by making use of internal or theoretical actions and operations, supported by external or practical actions and operations. The meaning of the generalization–empirical–product of this process is crystallized or deposited in some representation–definition, proposition, method–. Next, the empirical generalization obtained is subjected to a formalization process in which theoretical actions and operations are supported by theoretical principles through deduction or mathematical demonstration, to ensure that the constructed knowledge is valid and change its category from empirical to theoretical generalization.

The deductive way has been used in the teaching of mathematics with greater insistence, but here we are interested in testing the inductive way, considering first, that it favors the regulation of the process of formation of IAOT in students; and second, that it may be closer to the real process of construction of mathematical knowledge.

Consequently, from DPTCMO that we are formulating, the inductive way in the cognitive processes on MO, allows to establish three stages for its realization according to **Figure 3**:

1. The empirical stage in which generalizations are produced from internal or theoretical actions and operations supported by external or practical actions and operations. The generalizations resulting from this stage are always empirical.

[5] DPTCMO means didactic perspective for the teaching of concepts about mathematical objects.

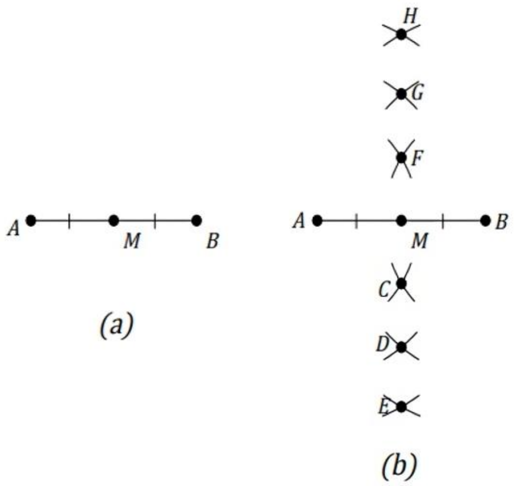

Figure 4. Approach to representation of object (Source: Authors' own elaboration)

2. The formal or theoretical stage: in this stage, empirical generalizations undergo a formalization process in which a change of knowledge category can occur, from empirical to theoretical. The internal or theoretical actions and operations necessary for the realization of this process are based on theoretical principles–as a logical consequence derived from mathematical principles–. In other words, it is about mathematical deduction or demonstration, whose function in the process of construction of mathematical knowledge is to assure or validate empirical knowledge.
3. Of application of the constructed knowledge, this knowledge together with the previous validated results, applied to new situations can give rise to new knowledge: Within mathematics, generalizing new features for the concept on the newly formed MO, making possible its development; or producing a new generalization. In other disciplines, new generalizations may be produced to solve some problem of practical activity, among other possibilities.

EXEMPLIFICATION OF IMPLEMENTATION OF FORMULATED DIDACTIC PERSPECTIVE

Concept on Object

Empirical stage–Activities

Activity 1: In part a in **Figure 4**, is given a segment AB and its midpoint M. With the compass determine the location of several points on either side of AB that satisfy the property of being equidistant from the ends of the segment. Does the midpoint M satisfy this property?

Activity 2: In part b in **Figure 4**, some points equidistant from the ends of AB are represented, with the help of a ruler join those points–one by one–. What kind of line do you think it is? What angle–can you measure it–does it form with the segment AB?

In this case, the above questions are only intended to guide the analysis of the object of study, with the support of different practical operations. As a product of this analysis, it is possible to notice some outstanding features: apparently, a straight line is formed by joining the equidistant points of the ends of the segment, it forms a right angle with the segment and passes through its midpoint. Regarding the action of abstraction, in this case there is nothing to be dispensed with, so the above features are abstracted in their entirety. By checking whether these features can be extended to new similar situations, they can be generalized as essential features common to all similar objects studied.

In synthesis, we can crystallize or deposit the meaning of the knowledge acquired about such objects in some form of language, that is, in a representation. In other words, we can formulate a definition–a written verbal expression–in which the substantial features of the defining object are listed:

> The equidistant points of the ends of a given segment are on the perpendicular that passes through the midpoint of the segment. Moreover, this special straight line can be called by the term mediatrix of a segment.

We must be aware that MO only exist at the conceptual level, as has been pointed out, it is the mental image that we form on the basis of its internal or essential and common features, the same ones that appear in the representation. But the written verbal expression we have chosen to crystallize the meaning of the knowledge built on the mediator is one of its material representations. Therefore, MO should not be confused with its representation.

Formal stage–Activities

The previous generalization has the category of empirical, since even when it has been produced through IAOT, these were supported by operations of a practical type–visual perception of the line that joins the points as a straight line, measurement of the angle it forms with the segment–. And although so far it is not a certain knowledge, there is a strong suspicion that it may be true. Therefore, it is necessary to move on to the formal stage of this process of knowledge construction in which we will seek to

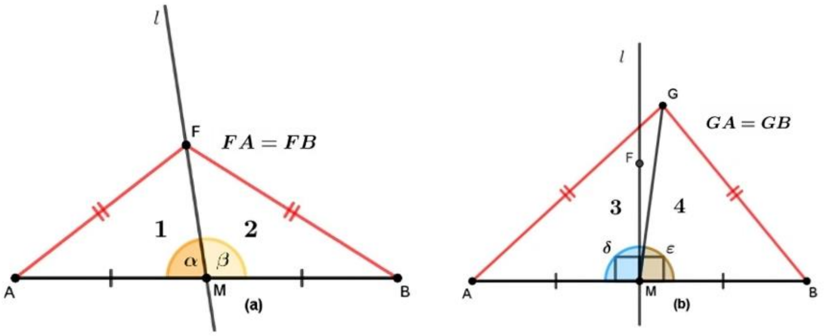

Figure 5. Representation of situation (Source: Authors' own elaboration)

obtain the same generalization as before, but without the aid of practical operations, but with the support of theoretical principles. This process can be achieved through mathematical deduction or demonstration. Thus, the previous generalization will change its category from empirical to theoretical; it will become part of the set of principles of the theory and subsequently, it can be used in processes of formalization of mathematical knowledge.

Activity 3: A first objective in this activity is to know if the line that passes through the points–located by practical operations–equidistant from the ends of the segment AB is indeed a straight line. The analysis must be guided by our suspicion that the mentioned points are on a line.

We know that there is always a line that passes through two given points, so that, if a line is drawn through two of the points that satisfy the property of being equidistant from the ends of the segment, it would only be necessary to prove that the others are also on that line.

Based on part a in **Figure 5**, we realize that by tracing the line l through M and F, two triangles Δ1 and Δ2 are formed, which are congruent, since they have their three sides, respectively equal; consequently their angles α and β are equal because they are corresponding parts of congruent figures. Moreover, if these are collinear, $\alpha + \beta = 180$, then $= \beta = 90°$. This guarantees that the line l is perpendicular to the segment AB at its midpoint M.

In part b in **Figure 5**, we have already recorded the fact that $l \perp AB$and that it passes through M. Now, as the second objective of the activity, we are interested in proving that the other points with the property of being equidistant from the ends of AB are also in L; to perform the analysis we will take any point with that property and try to prove the following proposition:

Proposition (A): If $l \perp AB$ at its midpoint M, G is any point with the property of being equidistant from the ends of the segment AB. Then, G is in l.

Before performing the proof, it is necessary to open a space to discuss two principles of mathematical theory about any logical proposition. We begin by saying that given any logical proposition (*P*), there are three possibilities regarding its truth value: a) *P* is true; b) its negation is true, or c) both are true at the same time

$$\left[\begin{array}{lll} \text{a)} & P & (V) \\ \text{b)} & \sim P & (V) \\ \text{c)} & \{P \wedge \sim P\} & (V) \end{array}\right. \longrightarrow \begin{array}{lll} \text{a)} & P & (V) \\ & & \\ \text{b)} & \sim P & (V) \end{array}$$

But on the basis of the following principle:

Non-contradiction: Mathematical theory does not admit any contradiction in its discourse. The third of the above options has to be excluded–third excluded–, since it represents a contradiction, leaving the first two possibilities.

In short, this is expressed in the following principle:

Excluded third: Given a logical proposition, it is either true or its negation is true.

Such principles are the basis of the method of demonstration called reductio ad absurdum. This consists in assuming as true the negation of the thesis ($\sim T$) corresponding to the proposition to be proved; but if this assumption leads us through logical reasoning to a mathematical contradiction, we must be aware that we will have to discard this assumption ($\sim T$) to avoid the contradiction, that is, it is not possible to sustain it as true; consequently, based on the principle of the excluded third party, only the option that T is true remains. We will use this method of demonstration to perform the formal proof on the validity of the empirical generalization obtained above. That is, we will assume as true the negation of the thesis corresponding to proposition (A). $\sim T$: (G) is not in l.

And we will make the analysis based on part b in **Figure 5**. From the point G located outside l we plotted the distances to the ends of AB and the line GM. With this, two triangles Δ3 and Δ4 are formed, which turn out to be congruent by having their three sides, respectively equal; consequently their angles $\delta = \varepsilon$ being corresponding parts of congruent figures. Moreover, their sum $+\varepsilon = 180°$because they are collinear, therefore $\delta = \varepsilon = 90°$, which means that $GM \perp AB$ in M. This constitutes a contradiction with the principle that states that through a point (M) on a straight line (AB) passes a single perpendicular to that line.

As said, the assumption of ~T: (G) *is not in* l as true causes a contradiction and it is not possible to hold it as true. Consequently, based on the excluded third party principle, T: (G) is in l is true, which concludes the proof.

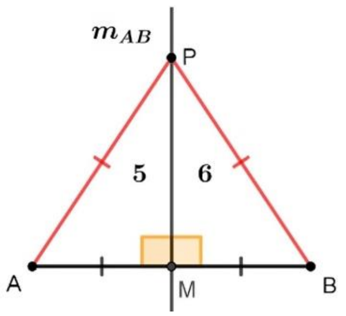

Figure 6. Mediatrix of segment AB (Source: Authors' own elaboration)

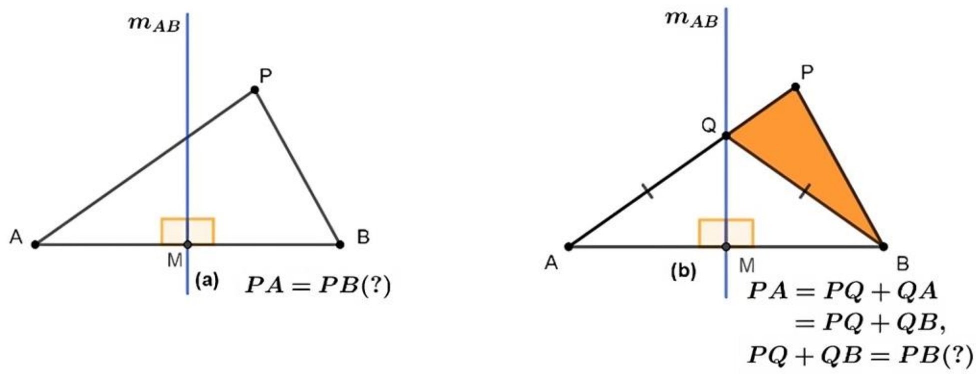

Figure 7. Representation & analysis of perpendicular bisector of segment AB (Source: Authors' own elaboration)

Similar proofs could be performed for each point with such property, but this is no longer necessary, since in the proof performed any point with the property was considered and this allows to generalize the feature of belonging to the line l for points equidistant from the ends of the segment AB. In summary, an expression can be formulated to crystallize the meaning of the constructed knowledge: *The points equidistant from the ends of a given segment are on the perpendicular that passes through the midpoint of the segment*. Moreover, this special line can be denoted by the term mediatrix of a segment.

Let us emphasize two aspects. On the one hand, the generalization produced in the formal stage, has the same form as that obtained in the empirical stage, but differs in one aspect, it now has the category of theoretical knowledge, since it was obtained in a formalization process, by means of mathematical deduction or demonstration. This change exemplifies the passage from the concrete to the abstract in relation to the process of knowledge construction, according to the Marxist point of view. On the other hand, when reviewing the statement of the previous immediate generalization, even though it already has the category of theoretical knowledge, we notice that, although the equidistant points are in the mediatrix, we do not know if all its points have this property. And this is an aspect that can be of interest in the construction of knowledge about the perpendicular bisector of a segment. To find out, we propose the following activity:

Activity 4: We want to know if all the points of the perpendicular bisector have the feature of equidistance to the ends of the segment. This can be solved by proving that any point on the perpendicular bisector has this property.

In **Figure 6** is represented a segment AB, its midpoint M and its perpendicular bisector m_{AB}. If any point (P) of this line is taken and its distances to the extremes A and B are plotted, the right triangles Δ5 and Δ6 are formed, whose hypotenuses are the distances that we are interested in knowing if they are equal. One way to investigate this is to try to know if these triangles are congruent, because in that case the homologous parts–corresponding–are equal.

Without much difficulty, both teachers and students to whom this proposal is addressed can indeed conclude that these triangles are congruent and finally that the distances PA and PB are equal. Similar to the previous case, it will no longer be necessary to perform further tests for other points, since it was done for any point belonging to m_{AB}. In summary, we can use a written verbal representation to deposit the meaning of the constructed knowledge: *all points on the perpendicular bisector of a given segment are equidistant from the ends of the segment*.

The fact that both teachers and students successfully complete this test represents a development in their knowledge about this object, since they will have managed to incorporate a new feature to the mental image they had formed, that is, to their concept of the perpendicular bisector. Another factor for the development of a concept is related to the change in the category of knowledge. In order to continue with the development of the concept of the mediator, the following activity is proposed:

Activity 5: Analyze the following situation represented in **Figure 7**: In the plane is given a segment AB and its bisector m_{AB}, it is interesting to know if there is any point in the plane outside m_{AB}that can have the feature of equidistance to the ends of AB.

In part a in **Figure 7** we have taken any point in the plane outside m_{AB} and plotted its distances to the ends of segment AB. So finding out whether P can have the feature of the points on the perpendicular bisector is equivalent to investigating whether PA and PB can be equal.

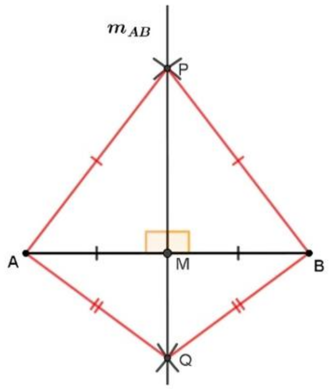

Figure 8. Representation of line of perpendicular bisector with ruler & compass (Source: Authors' own elaboration)

In part b in **Figure 7**, the point Q of intersection between PA and the m_{AB} has been highlighted and the segment QB has been drawn, thus forming the triangle PQB. With the aid of this representation the following valid relation can be established:

Relation (1): Segments PQ, QA are collinear, then; $PA = PQ + QA \cdots (1)$. It is also satisfied that $Q \in (m_{AB})$, then $QA = QB \cdots (2)$. Therefore, $PA = PQ + QB \cdots (3)$. Consequently, investigate whether: $PA = PB(?)$ is equivalent to knowing whether: $PQ + QB = PB(?)$.

In this regard, it is known that the shortest distance between two points is the line segment that joins them. This principle applied to any triangle, gives rise to the modern expression called triangle inequality: the sum of two of any of its sides is always greater than the third; applying this knowledge to the case of the triangle PQB of part b in **Figure 7**, we have:

Relation (2): In triangle PQB it is satisfied that $PQ + QB > PB$ and $PA = PQ + QB \cdots (3)$. Then, $PA > PB$.

This shows that any point P of the plane outside m_{AB} is not equidistant from the ends of AB; this allows us to generalize this feature for all points of the plane outside m_{AB} and in synthesis to formulate an expression that crystallizes the meaning of the new knowledge constructed. We could simply express: *there is no point of the plane outside the perpendicular bisector that is equidistant from the ends of the segment*. But first it was proved that all points on the perpendicular bisector have the feature of equidistance. With these two results a conclusion can be formulated: *All points in the plane that are equidistant from the ends of a segment are on the segment's perpendicular bisector*.

This is another development factor for the concepts, it has to do with what we call generalization volume: in this case, it has been possible to generalize the feature of belonging to the perpendicular bisector to any point of the plane that has the property of equidistance to the ends of the given segment.

Concept of Operations

Empirical stage–Activities

Up to this point of PCF on MO called the bisector of a segment, we can consider that the concept that we have formed of this object, is acceptable; that is, our mental image, reflects its features: it is perpendicular and passes through the midpoint of the segment, it contains all the points of the plane that are equidistant from the ends of the segment. But we still do not know how to draw it or construct it. Therefore, with the dual purpose of covering this absence, and exemplifying PCF on operations, we will pose the following activity:

Activity 6: Given any segment AB (**Figure 8**), trace its bisector m_{AB} with ruler and compass.

Basis of activity orientation: Considering the knowledge built above, it can be established that: *The bisector of a segment is a perpendicular line passing through its midpoint. It contains all the points of the plane that are equidistant from the ends of the segment*. A single line passes through two given points.

Based on the above, the following question was asked: If the location of two points equidistant from the ends of the given segment were known, would they be on the perpendicular bisector?

After considering the previous knowledge, as a product of the analysis, the need to locate two points equidistant from the ends of the segment AB is highlighted, since the line that passes through them is the mediatrix sought. Thus, you will find an effective procedure. Then you can abstract the essential operations required to draw the perpendicular bisector. Then it is checked if this set of operations–procedure–can be effective for any segment; if so, then it will be generalized, and we are in front of a PCF of operation, which unlike the concept on object, instead of writing a definition, the essential and common operations in all cases–method–to draw the perpendicular bisector of a given segment are described; in short: method to draw the perpendicular bisector of a given segment (with ruler and compass). Locate two points equidistant from the ends of the segment. 2. The straight line passing through these points is the requested perpendicular bisector.

Formal stage–Activities

At this point, it can be said that the generalization–method–produced is empirical. To change its category to theoretical knowledge, teachers or students are required to show formal arguments–based on theoretical principles of mathematics–that assure that with the described procedure the perpendicular bisector of a segment is always obtained. For example, based on **Figure 8**, the following result is established and obtained:

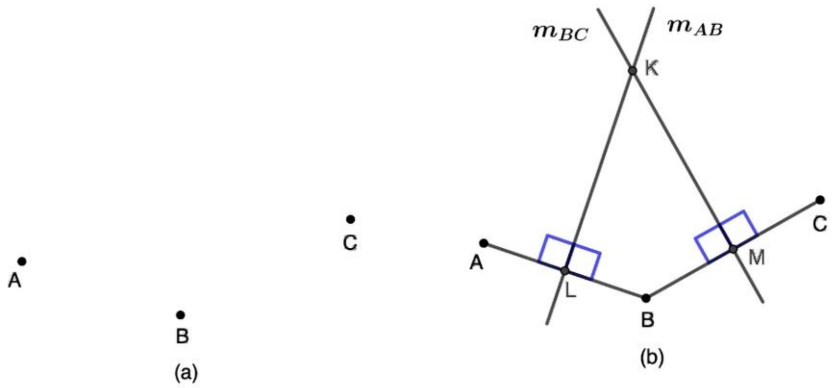

Figure 9. Representation of point equidistant from three other non-collinear points (Source: Authors' own elaboration)

Proposition (B): P and Q are equidistant from the ends of AB, two points determine a single line, moreover, contains all points equidistant from the ends of. Then, the line through P and Q is $m_{AB} \cdots (*)$. The result of $(*)$ is the formal proof that the line through P and Q is the requested perpendicular bisector and its justification did not require any practical operation.

Applications of Constructed Knowledge: Inside Mathematics & Practice Problems

Activity 7

The problem of finding a point equidistant from three other non-collinear given A, B, and C (part a & part b in **Figure 9**). It can be seen as a PCF of operation: method for locating a point equidistant from three other non-collinear dice.

If we pose it to students who have recently formed the concept of mediatrix of a segment, then, in the analysis of the situation they will try some procedures (P_1, P_2, P_3, $\cdots$) that will not lead them to the solution. This is a good moment to guide their activity, pointing out that sometimes it is convenient to breakdown a complex situation into simpler ones to facilitate the task of analysis; that is, if in this case the requirement is to find a point equidistant from three others, we can first suggest finding a point equidistant from only two. With this suggestion and the knowledge they have built up they will be more likely to solve the problem posed. Thus, they will find a procedure P_k that is effective, as is the trace of the perpendicular bisectors m_{AB} and m_{BC} whose intersection determines the point (K) equidistant sought. Then, by abstraction, the erroneous procedures are dispensed with and the procedure P_k is adopted. Then it is checked if this procedure can be effective for any trio of points that are non-collinear; if so, then it will have been generalized, and we are before an operation PCF that, unlike the concept on object, instead of writing a definition, the essential and common operations in all cases–method–are described to determine the position of the point equidistant from other three given non-collinear ones, in synthesis; one has the following method: Method to locate a point equidistant from three other given non-collinear A, B and C (with ruler and compass):

1. Draw the perpendicular bisector of BC.
2. Draw the perpendicular bisector of BBC.
3. The intersection K of both perpendicular bisectors is the point we are looking for.

At this point in the development of the knowledge construction process, it can be said that the generalization–method–produced is empirical. To change its category to theoretical knowledge, students are required to show formal arguments–based on theoretical principles of mathematics–that assure that with the described procedure a point with the requested property is always obtained. For example, based on part b in **Figure 9** the following result is obtained: since $K \equiv (m_{AB} \cap m_{BC})$, then K is on both medians. Furthermore, if K is on the m_{AB}mediatrix, then $KA = KB$, analogously; if K is on the m_{BC} mediatrix, then $KB = KC$. Therefore, $KA = KB = KC \ \cdots (**)$. The result $(**)$ is the formal proof that K is equidistant from the three given points A, B, and C and its justification did not require some practical type of operation. If any student insists on defending the validity of his generalization, supported by practical operations, it is an indicator that his concept–method–still remains in the category of empirical knowledge.

Activity 8

A new problem related to the result of the previous situation can be posed, which consists of knowing if it is possible to make a circumference pass through three given non-collinear points. And if possible, to know how many can pass through these points. The problems described in activity 7 and activity 8 can be considered as activities corresponding to the passage from the abstract to the abstract of validated knowledge about the mediating object; that is, they are applications of knowledge, within mathematics. But as it has been said, there can also be applications outside, in the practical activity, as proposed in activity 9, which is proposed as an activity for independent work.

Activity 9

The neighbor ordered his blacksmith to install a sliding gate. But the gate is too wide, and when it opens, it can collide with the wall on the side. To avoid the problem, it is necessary to bend a part of the rail on which it will run, so that it has the shape of a circular arc. **Figure 10** shows the outlines of the project, where you can see the location of three points A, B, and C, which should

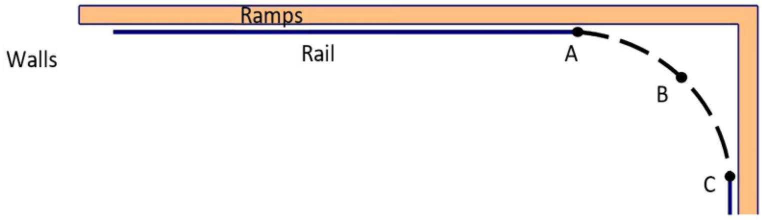

Figure 10. Representation of study situation (Source: Authors' own elaboration)

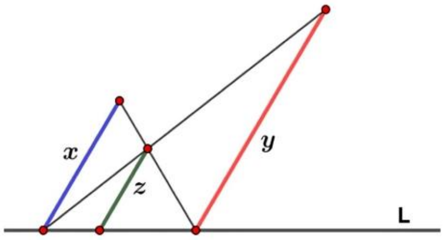

Figure 11. Geometric representation of relation of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ segments (Source: Authors' own elaboration)

determine the magnitude and curvature of the circular arc with which you could make the mold to shape that part of the rail. The blacksmith asks for your help to get the radius of the circular arc that passes through the three marks.

Concept of Relation

Empirical stage

Two parallel segments (see **Figure 11**) $\boldsymbol{x}, \boldsymbol{y}$ have each one an end on a straight line $\boldsymbol{L}$. If straight lines are drawn from the ends of these segments that remain outside, to the ends on the line, and if by the point of intersection of these lines the segment $\boldsymbol{z}$ parallel to $\boldsymbol{x}, \boldsymbol{y}$ is drawn, find a relation between the segments $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$.

In a convenient computational medium, a representation can be designed that is feasible to perform transformations[6] on it, with the purpose of breaking down the features in the different conditions of the given situation. As a product of the analysis, the existence of non-essential features is identified: the segment $\boldsymbol{z}$ does not change its length if $\boldsymbol{L}$ changes its direction, nor if $\boldsymbol{x}, \boldsymbol{y}$ change their distance from each other. However, there is an essential feature: the value of $\boldsymbol{z}$ changes if $\boldsymbol{x}$ or $\boldsymbol{y}$, or if both change.

By *abstraction*, we can dispense with the non-essential features and emphasize the essential feature: the existence of a relation in which $\boldsymbol{z}$ depends only on $\boldsymbol{x}, \boldsymbol{y}$. To investigate whether in other similar situations this feature prevails, representations of new similar situations must be reproduced, and the necessary transformations made on them to perform this analysis. It is important to realize that each transformation on the representation elaborated in the computational environment represents a new similar situation and then it can be said by *generalization* that the abstracted essential feature can be extended to any trio of segments that meet the conditions of the problem posed. In short, provided the conditions of parallelism are met, there exists a relation that establishes that the value of $\boldsymbol{z}$ is only a function of $\boldsymbol{x}, \boldsymbol{y}$.

What does this relation look like? At this point we can only say that it is a qualitative relation that establishes the dependence of $\boldsymbol{z}$ as a function of $\boldsymbol{x}, \boldsymbol{y}$. In its quantitative aspect, it is impossible to know, it could contain additions, differences, products, quotients or a mixture of these operations. In accordance with the approach to PCF, the empirical stage cannot go any further in this case, since it is difficult to know what this relation will be. But in the second *formalization* stage we will be able to obtain exactly what this relation is.

Formalization stage

At this stage, IAOT can only be supported by theoretical principles. So the analysis begins with the support of **Figure 12**, which contains a representation of the situation described. In this one it is detected that the condition of parallelism between the segments $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, gives rise to the similarity of two pairs of triangles $\Delta1 \sim \Delta2$ see part a in **Figure 12** and $\Delta3 \sim \Delta4$ *s*ee part b in **Figure 12**, whose proportionality between their homologous sides can be expressed, as follows:

$$\left\{\begin{matrix} \Delta1 \sim \Delta2: \frac{n}{z} = \frac{m+n}{x} \cdots (I) \\ \Delta3 \sim \Delta4: \frac{m}{z} = \frac{m+n}{y} \cdots (II) \end{matrix}\right\}$$

Adding (I) and (II) we get:

$$\left\{\begin{matrix} \frac{m}{z} + \frac{n}{z} = \frac{m+n}{x} + \frac{m+n}{y} \\ \frac{m+n}{z} = \frac{m+n}{x} + \frac{m+n}{y} \end{matrix}\right\} \longrightarrow \frac{1}{z} = \frac{1}{x} + \frac{1}{y} \cdots (III)$$

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[6] If this resource is not available, students will elaborate different representations on paper to carry out the described transformations, with the purpose of performing the corresponding analysis.

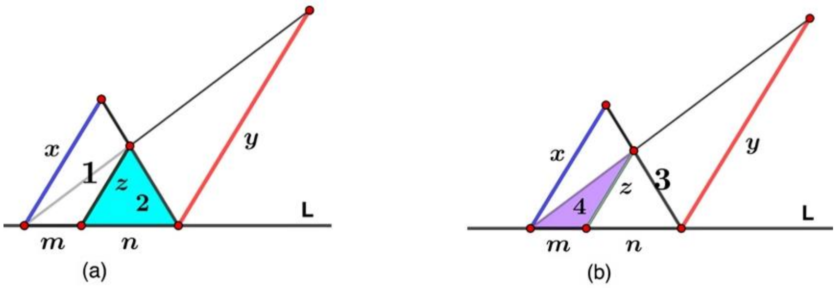

Figure 12. Representation & identification of pairs of similar triangles (Source: Authors' own elaboration)

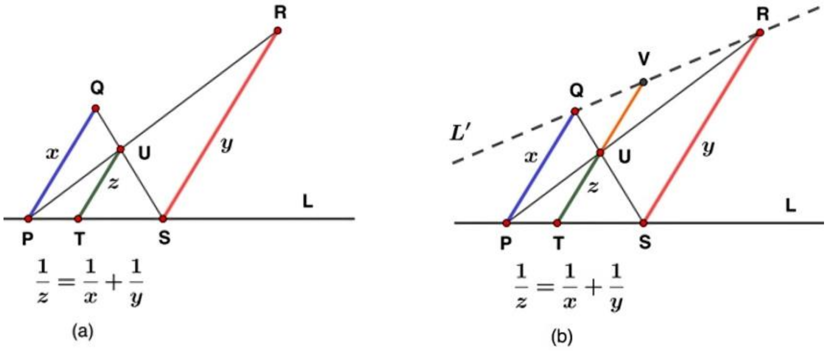

Figure 13. Representation of half harmonic mean (Source: Authors' own elaboration)

And this relation–an essential feature–can be generalized to any similar situation if the conditions of parallelism between the segments involved prevail.

The empirical generalization produced in the first phase of PCF, only referred to the existence of a relation in which one variable depended on two others; that made the mental image we formed of this concept of relation too vague perhaps; but after the formal stage, that mental image is clarified, we already know how it is; it is a quantitative relation, because knowing the values of $\boldsymbol{x}$ and $\boldsymbol{y}$, the value of $\boldsymbol{z}$ is calculated without any doubt. This is an invariant relation, not because the values of the variables involved do not change, what does not change is the way in which they are related. Of course, as long as the condition of parallelism between the segments $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ holds together with the conditions described in the initial situation.

On inquiring in the mathematical literature whether there is any known way of referring to such a relation, it was found that, by its form, it corresponds to an expression representing $\boldsymbol{z}$ as *half the harmonic mean*, between $\boldsymbol{x}, \boldsymbol{y}$.

It is known that three given quantities $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are in–or form a–harmonic progression if their reciprocals are in–or form an–arithmetic progression.

According to this, if $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, is a harmonic progression, then $\boldsymbol{b}$ is the harmonic mean of $\boldsymbol{a}$ and c, i.e., $\boldsymbol{b} = \boldsymbol{H}(\boldsymbol{a}, \boldsymbol{c})$. Furthermore, if $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ form a harmonic progression, then their reciprocals $\frac{\mathbf{1}}{\boldsymbol{a}}, \frac{\mathbf{1}}{\boldsymbol{b}}, \frac{\mathbf{1}}{\boldsymbol{c}}$ form an arithmetic progression in which $\frac{\mathbf{1}}{\boldsymbol{b}}$ is the arithmetic mean of $\frac{\mathbf{1}}{\boldsymbol{a}}$ and $\frac{\mathbf{1}}{\boldsymbol{c}}$, i.e., $\frac{\mathbf{1}}{\boldsymbol{b}} = \boldsymbol{A}\left(\frac{\mathbf{1}}{\boldsymbol{a}}, \frac{\mathbf{1}}{\boldsymbol{c}}\right)$.. Then, if $\frac{\mathbf{1}}{\boldsymbol{a}}, \frac{\mathbf{1}}{\boldsymbol{b}}, \frac{\mathbf{1}}{\boldsymbol{c}}$ form an arithmetic progression, their arithmetic mean is $\frac{\mathbf{1}}{\boldsymbol{b}} = \frac{\frac{\mathbf{1}}{\boldsymbol{a}}+\frac{\mathbf{1}}{\boldsymbol{c}}}{\mathbf{2}}$,, that is:

$$\frac{\mathbf{2}}{\boldsymbol{b}} = \frac{\mathbf{1}}{\boldsymbol{a}} + \frac{\mathbf{1}}{\boldsymbol{c}} \cdots (IV)$$

This last expression (IV), even though it was derived from the expression for the arithmetic mean, in it b can be identified as the mean of the initial harmonic progression $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ from which this analysis started.

Now, when comparing expressions (III) and(IV), a certain similarity is identified between them, which can be increased by performing some theoretically allowed operations. Thus it is obtained that $z = \frac{b}{2} \cdots (V)$. According to this analysis, z in expression (III) represents *half of the harmonic mean*, between $\boldsymbol{x}, \boldsymbol{y}$ and with this denomination, we will refer to this expression from now on.

In order to promote some development of the concept of *half of the harmonic mean*, the following set of activities is proposed. To do so, it is suggested to take some time to analyze part a in **Figure 13**, to visualize the conditions under which the relation that identifies $\boldsymbol{z}$ as half of the harmonic mean, between $\boldsymbol{x}, \boldsymbol{y}$, is fulfilled. This will facilitate your answers to the following questions:

1. If in part b **Figure 13**, the line L' is drawn through the points Q and R–extremes of $\boldsymbol{x}, \boldsymbol{y}$–; also the segment $\boldsymbol{w}$ parallel to $\boldsymbol{x}, \boldsymbol{y}$ at the point U. Does it seem to you that $\boldsymbol{w}$ has the same conditions as $\boldsymbol{z}$? It is too obvious.
2. How is it related to $\boldsymbol{x}, \boldsymbol{y}$? $\frac{\mathbf{1}}{\boldsymbol{w}} = \frac{\mathbf{1}}{\boldsymbol{x}} + \frac{\mathbf{1}}{\boldsymbol{y}}$.
3. Is the sum $\boldsymbol{z} + \boldsymbol{w}$, equivalent to the segment TV? It is known that only a single parallel can pass through a point (U) outside a line. If the segments $\boldsymbol{z}$ and $\boldsymbol{w}$ are both parallel to $\boldsymbol{x}$, passing through $\boldsymbol{U}$, it means that they are on the line parallel to $\boldsymbol{x}$ passing through $\boldsymbol{U}$; in other words, they are collinear, and their sum $\boldsymbol{z} + \boldsymbol{w}$ is equal to the segment TV.

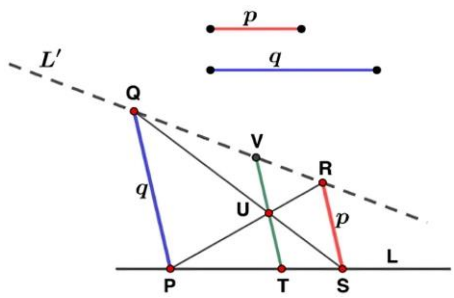

Figure 14. Geometric representation of method for finding harmonic mean (Source: Authors' own elaboration)

4. What does the sum of the segments $\boldsymbol{z}+\boldsymbol{w}$ represent? If each of the segments $\boldsymbol{z}$ and $\boldsymbol{w}$ represent half of the harmonic mean, between $\boldsymbol{x}, \boldsymbol{y}$, then their sum, which is equal to the segment TV represents the harmonic mean, between $\boldsymbol{x}, \boldsymbol{y}$.
5. Then, if in part b in **Figure 13** segment TV represents the harmonic mean, between $\boldsymbol{x}, \boldsymbol{y}$, could you give a method (with ruler and compass) to find the harmonic mean, between any two other segments given $\boldsymbol{p}$ and $\boldsymbol{q}$?

Method to find (with ruler and compass) the harmonic mean of any two given quantities $\boldsymbol{p}$ and $\boldsymbol{q}$ (see **Figure 14**).

- Draw two parallel segments of magnitudes $\boldsymbol{p}$ and $\boldsymbol{q}$, respectively, with one of their ends on any line L.
- Through the remaining ends of $\boldsymbol{p}$ and $\boldsymbol{q}$, draw the line $\boldsymbol{L}'$. Draw, in addition, the segments that cross the ends of $\boldsymbol{p}$ and $\boldsymbol{q}$, determining their intersection U.
- Through the intersection point U, draw a line parallel to $\boldsymbol{p}$ and $\boldsymbol{q}$, whose intersections with L and $\boldsymbol{L}'$ are T and V, respectively.
- The segment TV joining these intersections is the harmonic mean of the given quantities.

As a summary, the different symbologies used to represent this concept of the harmonic mean of two given quantities are presented:

If $(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t})$are in (or form a) harmonic progression. Then $\boldsymbol{s}$ is the harmonic mean of $\boldsymbol{r}$ and $\boldsymbol{t}$. Symbolically: $\mathbf{s}=\mathbf{H}(\mathbf{r}, \mathbf{t})$**.**

$$\frac{\mathbf{2}}{\boldsymbol{s}}=\frac{\mathbf{1}}{\boldsymbol{r}}+\frac{\mathbf{1}}{\boldsymbol{t}} \cdots (A)$$

or equivalent:

$$\boldsymbol{s}=\frac{\mathbf{2}\boldsymbol{rt}}{\boldsymbol{r}+\boldsymbol{t}} \cdots (B)$$

6. In **Figure 14**, we can visualize some features: the quadrilateral $PQRS$ is a trapezoid since it has parallel sides–its bases–, the segment TV, which we have identified as the harmonic mean of the segments Q and RS passes through the intersection of the diagonals of said trapezoid. Could you formulate or synthesize a generalization that expresses the relation between the segment TV and the bases of the trapezoid?

The requested expression would have the following form:

> In any trapezoid, the segment parallel to the bases passing through the intersection of the diagonals, and bounded by the nonparallel sides, is the harmonic mean of the bases of that trapezoid.

7. A variation on the method for constructing the harmonic mean of two given quantities arises in the following situation: If three quantities $\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{m}$ are to form a harmonic progression, but only $\boldsymbol{j}$ and $\boldsymbol{k}$ are known, how can $\boldsymbol{m}$ be obtained to complete the harmonic progression?

Of course, a solution can come from the numerical way by using expression (A) or expression (B). But here, the interest consists in finding a solution from the geometrical point of view. Considering the meaning of the generalization formulated in the previously, and with the support in part a in **Figure 15** we could represent the known magnitudes as corresponding parts of a trapezoid; that is, $\boldsymbol{j}$ has to be one of its bases, $\boldsymbol{k}$ as harmonic mean has to be the segment that passes through the intersection of the diagonals parallel to $\boldsymbol{j}$ and $\boldsymbol{m}$–the unknown–, must be the other base.

An additional feature that has manifested itself in the realization of activities 5, 6, and 7 but that had not been emphasized, is that the point of intersection of the diagonals is the midpoint of the segment that represents the harmonic mean of the bases. And this feature is key in the solution because it allows to complete the trapezoid, as shown in part b in **Figure 15**, so that the segment $\boldsymbol{m}$ would be determined.

The procedure necessary to complete the harmonic progression $\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{m}$, when only $\boldsymbol{j}$ and $\boldsymbol{k}$ are known, is the following set of–essential–operations (see part a & part b in **Figure 15**):

- Plot $\boldsymbol{j} \parallel \boldsymbol{k}$ with one of its ends on any line L.
- By the remaining ends of $\boldsymbol{j}$ and $\boldsymbol{k}$, trace the line $\boldsymbol{p}$. Trace, in addition, the line $\boldsymbol{q}$–one of the diagonals of the trapezoid–joining the end of $\boldsymbol{j}$ outside $\boldsymbol{L}$ with the midpoint $\boldsymbol{M}$ of $\boldsymbol{k}$.
- Through the intersection $(\boldsymbol{q} \cap \boldsymbol{L})$, trace $\boldsymbol{r} \parallel \boldsymbol{j}$.
- The segment determined by $(\boldsymbol{q} \cap \boldsymbol{L})$ and $(\boldsymbol{p} \cap \boldsymbol{r})$ is $\boldsymbol{m}$, the segment that completes the harmonic progression $\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{m}$.

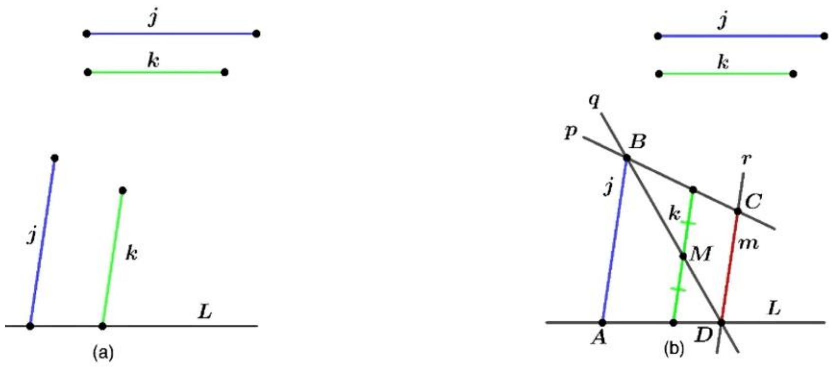

Figure 15. Alternate variation of method for finding harmonic mean (Source: Authors' own elaboration)

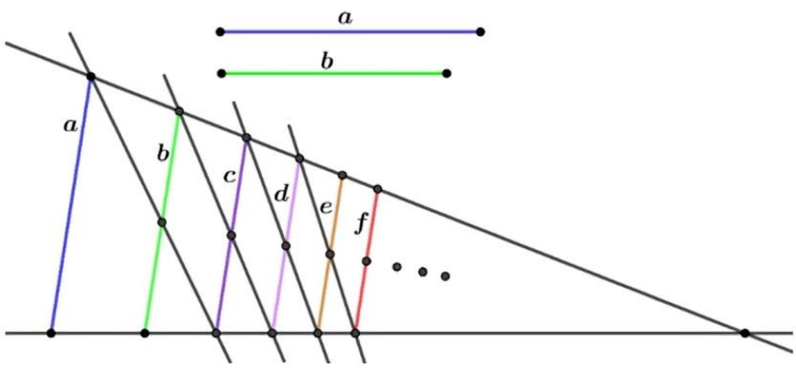

Figure 16. Representation of construction of harmonic progression (Source: Authors' own elaboration)

With this last procedure, one can keep adding terms to the progression; just keep in mind that, in any harmonic progression, if any three consecutive terms are considered, the middle one is harmonic mean of the other two.

Figure 16 shows a representation of how the harmonic progression $a, b, c, d, e, f, g, \ldots$, is constructed from the first two a and b terms.

Thinking about the development of the concept half of the harmonic mean, by way of the application of the acquired knowledge, we inquired in the extramathematical field, particularly in physical science. We found that for the case of a circuit composed of two resistors (R_1, R_2) in parallel, this scientific discipline has shown that the total resistance (R_T) of the circuit is calculated by the relation:

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} \cdots \cdots (*)$$

When compared to the expression representing half the harmonic mean of two given quantities:

$$\frac{1}{z} = \frac{1}{x} + \frac{1}{y} \cdots \cdots (III)$$

This suggests the possibility of establishing a complete analogy between them; this is achieved by means of an abstraction, that is to say, if we disregard the type of variables involved, only magnitudes related in the same way in both expressions will remain, so we can express: $\boldsymbol{R_T = z, R_1 = x, R_2 = y}$.. If we also remember that we know a method to obtain $\boldsymbol{z}$ from $\boldsymbol{x}$ and $\boldsymbol{y}$, then it will be possible to solve the electrical circuit with ruler and compass.

Method: On a straight line (L), the ends of two parallel scales[7] $(E_1 \parallel E_2)$ are placed, regardless of the direction and distance between them; the resistors R_1 and R_2 are placed on the scales, respectively, see **Figure 17**. From here, the method for constructing the half harmonic mean between R_1 and R_2 will result in R_T.

According to **Figure 17**, the red segment represents the magnitude of the resulting resistance R_T, which can be evaluated on any of these scales, by means of the trace parallel to L indicated by the red arrow. This study situation could be presented in a more complex form, for example, if one had an electrical circuit as more than two resistors in parallel, say four resistors. Physics has shown that, in cases like this, the relation between the resulting resistance R_T and those that make up the parallel circuit, is given by the following expression:

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}$$

[7] These scales must be graduated in the usual units of measurement to refer to the magnitude of electrical resistances (Ω, Ohm).

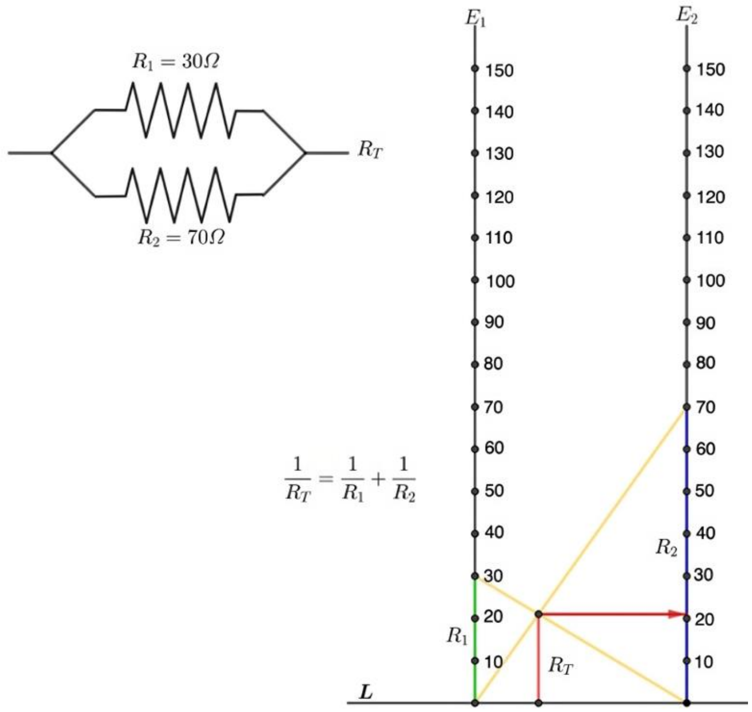

Figure 17. Representation of resultant in an electrical circuit & its relation to concept of half harmonic mean (Source: Authors' own elaboration)

Naturally the question arises whether the above method can be used in this case. But there will be no difficulty if a procedure such as the following is displayed:

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}, \frac{1}{R_T} = \frac{1}{R_{1-2}} + \frac{1}{R_3} + \frac{1}{R_4}, \frac{1}{R_T} = \frac{1}{R_{1-2-3}} + \frac{1}{R_4}, \frac{1}{R_T} = \frac{1}{R_{1-2-3-4}}$$

For its realization, it is suggested to make the necessary sketches of the circuit of study and to carry out practical operations on the representation to be made, by repeating the described method.

FACTORS INFLUENCING DEVELOPMENT OF MATHEMATICAL OBJECT CONCEPTS: DIDACTIC REFLECTIONS

In the field of mathematics teaching, in general, it is noticed that in PCF by students, it should not be expected–in the first confrontations with this task–that the concept is formed theoretically and with the highest degree of generalization; since this depends on the educational level, the prerequisites, among many other factors. Rather, what usually happens in practice is that their concepts show a permanent development, while it is not achieved that their mental image reflects the totality of the essential and common internal features, as well as the highest degree of generalization that corresponds to them.

Four factors influence this process of development: one has to do with the features of the object, another with the change of category of generalization, another with the volume of generalization and finally with the applications of knowledge in its passage from the abstract to the abstract and from the abstract to the concrete.

Features Factor

If the mental image of an MO, was formed on the basis of its external[8] features:

(1) some non-common or non-essential feature;

(2) incomplete common or essential features; but by performing new cognitive processes, it passes to another state in which such image, registers the complete common and essential features, we say that our concept about such MO has been developed.

At certain levels of education, and in the specific case of the formation of the concept of the perpendicular bisector of a segment, it is very common to observe that when students are asked to perform the activities of PCF, they form a mental image that reflects the features listed in the definition that appears in many texts: Mediatrix of a given segment, is a straight line perpendicular to the segment at the midpoint; but it does not reflect all the common and essential features as it does not contemplate the essential feature, that *"the mediatrix is the geometric locus of all points equidistant from the ends of the segment"*. In particular, complementing his mental image with this characteristic feature represents a development for his concept of the

[8] Let us designate *external features*, those features of MO that have been abstracted and generalized with the support of practical operations. Similarly, *theoretical features*, those that have been abstracted and generalized with the support of theoretical operations.

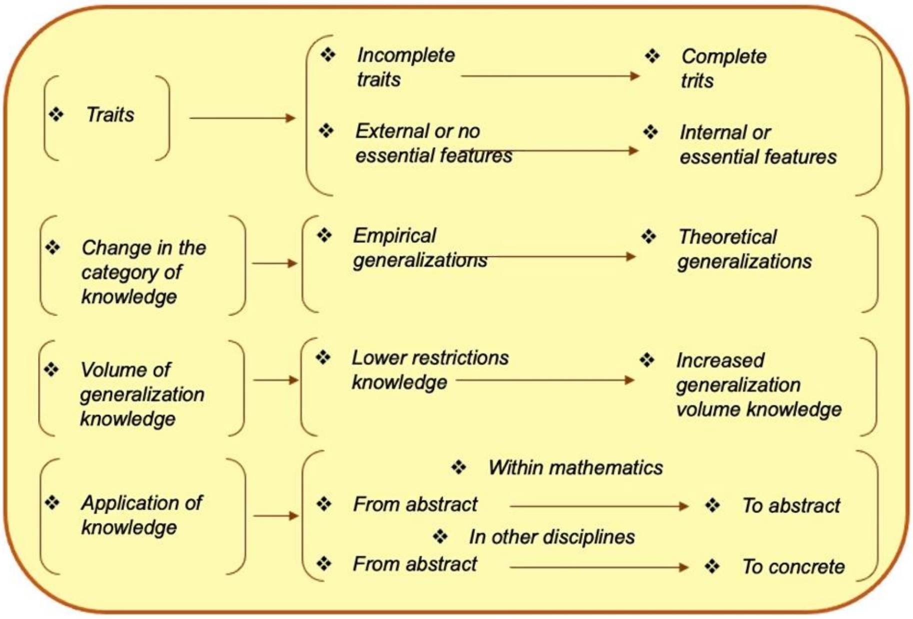

Figure 18. Factors in development of MO concepts (Source: Authors' own elaboration)

mediatrix. In general, the search for all essential and common features is a factor in the development of the concept of MO in particular.

Change of category of knowledge: In the example cited above lines, a way is described in how students can form a mental image of the mediatrix object, which registers the features of perpendicularity and passing through the midpoint of the segment, and we can even assume that they already performed the activities that allowed them to complement their mental image with the new characteristic feature, without having any proof that this is true; even if their image is correct, at this moment it corresponds to a concept of empirical character; because it is a generalization made on the basis of the external features–because they "see" that it is perpendicular, and because the equidistance is supported by practical resources, the measurements are taken with a graduated ruler or they are measured with the help of some software (GeoGebra). When they have gone to the internal features, essential and common to all mediatrixes–when it has been proven that it cannot be otherwise; then their concept will be developed and will have a theoretical character.

Another factor that influences the development of concepts has to do with the volume of generalization of the concept. To exemplify, let us think of a group of high school students who have already formed the concept of operation: method to locate a point equidistant from three other non-collinear given, it is clear that the volume of generalization of their concept, covers any trio of non-collinear points. But you, who are a teacher, have formed this concept in a wider volume of generalization, you know that this concept can be extended to the case of any three points, even collinear. You know that the volume of generalization of the concept that the students of the example now have, can be extended if you reduce the number of constraints (in the case under analysis, you can eliminate the constraint that the three points be non-collinear). You know that in this case the method also applies and that the equidistant point of the three given points is located at infinity, in fact; it is the ideal point shared by the two mediatrixes that in this case are parallel. Thus, the volume of generalization for this concept by the students can cover any trio of points, even collinear ones.

Now it is understood that while the students in our example may be satisfied with the volume of generalization reached in their concept formation process, the teacher knows that in order to develop their concept, by way of increasing the volume of generalization, he must necessarily work with them on some prerequisites–every ordinary straight line has a single ideal point; a family of parallels shares a single ideal point, which cannot always be covered at the educational level at which they are working. That is why we say that, in teaching, students' concepts manifest to be in a process of development. **Figure 18** shows the factors in the development of OM concepts.

Knowledge Applications

Knowledge applications constitute another factor in the development of concepts, because this activity adds meaning[9] to the knowledge built in PCF about an MO, in addition to making possible the construction of new knowledge. Thus, the application of knowledge of concepts such as the mediatrix of a segment generated new concepts about other MOs, and the corresponding solutions to some practical problems, as shown in **Figure 19**.

[9] Adding meanings to the knowledge built on an MO refers, in addition to the features of the object, to several aspects, such as: the method to build it, the new knowledge generated by relating it to other MOs–the links with other concepts–, the type of problems it can solve inside and outside mathematics, among other aspects.

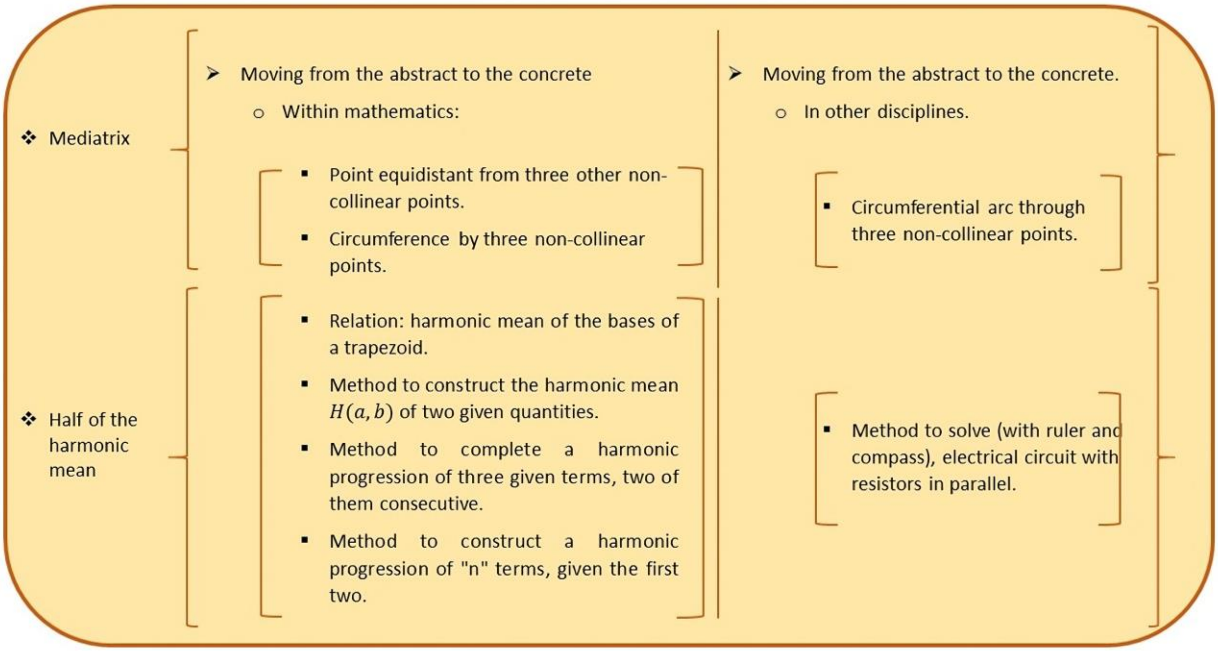

Figure 19. Concepts generated from study of mediatrix (Source: Authors' own elaboration)

CONCLUSIONS

To conclude from the research about process of formation and development of concepts about MO, we consider of interest both for in-service mathematics teachers and for future mathematics teaching professionals, to reflect on the following topics:

What is the difference between MO and concept about an MO? On the one hand, the concept about an MO is the knowledge about that object, which mankind has been able to abstract, generalize and synthesize after analyzing the object–or collection of similar objects–. This knowledge is a mental image that reflects the essential features of the object. Since this image is a product of abstraction, the concept about an MO is always abstract. On the other hand, MO has no material existence. Let us take the circumference as an example; let us suppose a practical activity in which some human being decides to drive a stake into the ground; tied to it, a rope of arbitrary size, at the end of which he ties a pointed object. He then moves the pointed object–with the string taut–around the stake, leaving the mark of its trajectory on the floor. When he reflects on this special trajectory, he decides by abstraction to dispense with the roughness of the ground, the width and depth of the trajectory, to consider that it is drawn on a completely flat surface and that it is a trajectory so thin that it has no thickness; he also dispenses with the stake and replaces it with a mark so small that it has neither length nor width–a point–, as well as the elasticity of the rope to ensure that each point of the curve is the same distance from the mark–center–. As a product of this abstraction a mental image is formed, which in synthesis reflects the following features of the circumference object: *circumference is a line containing points equidistant from another called center*. But such an object is impossible to find in concrete reality, it is clear that it can only exist on the plane of the abstract, at the conceptual level; and then we can say that MO and its concept are identical.

Some Difficulties in Teaching Mathematical Object Concepts

The adoption of the principle: to apprehend is to know, led us to seek guidance on the processes of knowledge construction. The theoretical approach reviewed at the beginning establishes that knowledge of the objects of reality starts from the sensory. If we think of following this approach to the process of knowledge in the teaching of mathematics, we immediately realize that MOs are not material objects, they are abstract–since they are the product of abstraction–; from there, we find a serious impossibility to try to know them from the sensory, that is, we cannot have a sensation, a perception or a notion of any MO. Therefore, knowledge of such objects is fundamentally conceptual; the use of representations is perhaps the only mitigating factor in this difficulty of a didactic nature. In teaching, representations of MOs are used in an attempt to make up for the absence of their material aspect, even if there is often the risk of confusing them with their representation.

Duval (1993) pointed out that such a difficulty of didactic character, recognized by the fact that most students confuse MOs with their representations, was due to a kind of inevitable cognitive paradox: On the one hand, in teaching, the idea prevails that the learning process–knowledge–of MOs is only possible through their semiotic representations, consequently, it is quite possible that students confuse MOs with their representations, since they can only have a relation with them.

On the other hand, Duval (1993) himself adds:

> This paradox is even stronger when teaching confuses mathematical activities–related to the use of procedures, algorithms, and in general the use of formal language for the acquisition of mastery of mathematical treatment–with conceptual activities, and also considers semiotic representations as secondary or extrinsic (p. 38).

And he ends by affirming: “only those who have built the concept on MO can recognize its different representations”.

How can this didactic difficulty be avoided? From our point of view, such didactic difficulty comes from the traditional way of conceiving the teaching of mathematics, that is, by insistently promoting the use of the deductive way for the realization of the process of knowledge of MO. This methodological way, which makes use of representations of MOs as a first approach to their knowledge, is the one known and traditionally used by most in-service teachers; this has made them think that such a process is only possible through the use of representations, without reflecting on any other possibility.

Our proposal to avoid such a difficulty of didactic nature is contained in this research, where we have formulated a didactic perspective for the teaching of concepts about mathematical objects (DPTCMO). This integrates two aspects:

1. Only those who have constructed the concept about MO can recognize its different representations (Duval, 1993).
2. According to the principle: To apprehend is to know, whoever wishes to learn mathematics must carry out processes of construction of mathematical knowledge, i.e., PCF on MO.

We exemplify its application, bearing in mind that apprehending mathematics by the inductive way is to carry out processes of construction of mathematical knowledge. In this way for PCF on MO, the learner has the opportunity to crystallize or deposit the meaning of the knowledge constructed on the object of knowledge, in some form of language, that is, in one or several forms of representation. Such an activity of knowledge construction, carried out in a systematic and consistent way, translates into the factor that can avoid confusing MO whose existence is abstract–mental image–and its representation of material character.

Derived from this proposal, the imperious need to be aware of the process of knowing about MOs arises, since it depends on this that teachers and future teachers can guide their students in this process.

As a next stage of the research, its experimentation with a population of pre-university and university teachers is planned. In this direction, it is assumed that a limitation of the present research lies in the lack of experimentation and the contrast between the theoretical-didactic elaboration and description and the empirical results that emerge from such experimentation.

Although it is not the objective of this research to influence curricular designs, the need to rethink the approaches to the mathematical content promoted in schools, particularly those related to the treatment of mathematical concepts, is raised.

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