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Description of the activated mathematical knowledge of the triangle concept in three empirical contexts

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ABSTRACT

The paper addresses concept formation processes of students in the field of geometry. More precisely, the paper deals with the questions of what knowledge do students activate about triangles in different contexts with different (digital) tools and furthermore what content-related meaning do they give to the concept of triangles? Methodologically, we use the descriptive framework of empirical theories for the analysis of our case study. The students develop different notion of the concept of triangles, using three empirical contexts like 3D pen, dynamic geometry software and pencil paper in an activity that allow students to construct the characteristics of triangles. In this article, we focus on how the students in our case study (further) develop and give meaning to the concept of triangle, which is central in geometry, in three empirical contexts; thus, engage in concept formation processes.

Keywords: concept formation processes, empirical contexts, 3D pen, triangle, geometry

INTRODUCTION

The description of the development of mathematical knowledge at interfaces with empiricism is a much-discussed topic in mathematics education. The use of materials has been the subject of much research, especially in elementary school mathematics. Furthermore, geometry teaching also focuses on a use of tools and manipulatives such as drawings. According to the KMK (2005), experiences with triangles belong to the competencies of "space and form" and "sizes and measurement". Related to the standards, the triangles activity in our study involves both students developing and using problem-solving strategies (e.g., systematic trial and error), as well as making drawings with tools, and comparing, measuring, and estimating quantities.

The discussion about empiricism and the use of manipulatives is intensified by technology and digital media which extend the range of possibilities of teaching and learning mathematics (graphically). An important aspect for this description is an appropriate representation of the relationship between mathematics and empiricism. This is necessary in view of current curricula, which assign an essential role to the reference to reality in mathematics teaching. In current math classes, a material-oriented and illustrative way of teaching and learning for various reasons has often been discussed (Thompson, 1994). After all, mathematics lessons should fundamentally teach students how to perceive and understand in a specific way phenomena of the world around us that should concern us all, from nature, society and culture (Winter, 1995). According to this, a reference to the physical world immediately surrounding us is important for mathematics teaching.

Hefendehl-Hebeker (2016) states that the concepts and contents of school have their phenomenological origins predominantly in the reality that surrounds us. For this reason, it is interesting how students develop their knowledge on these manipulatives in empirical contexts. In the sense of Winter (1995), there should be room for demonstrative-empirical working processes. A discussion about problems and potentials of representation (MacDonald et al., 2020)–especially in connection with digital tools and media–are often in the focus. There are already studies that deal e.g. with issues relating to 3D printing, 3D pen or dynamic geometry software (DGS) (for example GeoGebra) in mathematics lessons (e.g. Dilling & Witzke, 2020; Elschenbroich, 2018; Elschenbroich & Seebach, 2018; Kaenders & Schmidt, 2014; Ng, 2017; Ng et al., 2018; Panorkou & Pratt, 2016; Pielsticker, 2021). Thus, manipulatives that relate to this world are an essential element of teaching practice, as they can motivate, illustrate and justify mathematical relationships.

At the same time, learning of geometry in particular takes place with the help of manipulatives. Furthermore, learning geometry in early childhood has an important influence on mathematical learning (Clements & Sarama, 2011; Poon & Leung, 2016). Clements and Sarama (2011, p. 135) state that

> "geometry may be a gateway skill to the teaching of higher-order mathematics thinking skills."

Therefore, geometric figures in particular play a crucial role because

> "shape is a fundamental construct in cognitive development in and beyond geometry" (Clements et al., 2018, p. 9).

Also, triangles in particular are the most difficult figures for children to understand compared to circles and quadrilaterals (Maier & Benz, 2014; Satlow & Newcombe, 1998). At the same time, we find studies such as the recent work of Youkap (2021), which shows difficulties of students in giving characteristics of certain basic geometric figures that are conforming to the theory on these geometric figures. This shows the relevance of investigating how children develop the concept of triangle in the early phase and experience characteristic properties of triangles in activity in order to question previous teaching methods and to highlight advantages of learning geometry.

With this in mind, in this account with case study elements we review developments on the concept triangle of two students in three empirical contexts. We will describe students' knowledge developments using the empirical theories approach. Empirical theories provide a way to describe the development of student knowledge in these contexts. In short, an empirical theory is a theory that–like a scientific theory–describes and explains certain phenomena of reality. Under these premises, it seems necessary that mathematics education has viable description possibilities for dealing with empirical contexts.

This article looks at an example taken from a study conducted in a German elementary school showing how the students Anna and Jules (names changed) develop mathematical knowledge in empirical contexts and engages in concept formation processes using empirical objects. Relevant for our investigation is the interview conducted with the students and the activities that allow students to construct the characteristics of triangles. The focus is on the following research question:

What knowledge do students activate about triangles in different contexts with different (digital) tools (paper-and-pencil, 3D pen, and GeoGebra)? What content-related meaning do they give to the concept of triangles?

We therefore make use of the descriptive approach of empirical theories (Burscheid & Struve, 2020; Witzke, 2009) to describe our case and the exploratory processes in detail. Our case study is analyzed with an informal description in relation to the structuralist approach (Balzer, 1982; Sneed, 1971; Stegmüller, 1986). We refrain from a formal reconstruction, but use a fixed descriptive framework (Witzke, 2009).

THEORETICAL FRAMEWORK

A Descriptive Perspective

The basis of the analysis presented in this article is a constructivist view on learning processes. The main idea is that students construct their mathematical knowledge by themselves in interaction with their environment. Today it is generally acknowledged that students constitute their own mathematical knowledge in action and negotiation processes (Krummheuer, 1984). Learning in this sense is understood as an active process, dependent on individual experiential areas, constructing theories for an adequate cognition of certain phenomena (Burscheid & Struve, 2020). Students' knowledge can be described with the help of theories, usually empirical theories (Burscheid & Struve, 2020; Schiffer, 2019; Schlicht, 2016; Stoffels, 2020; Pielsticker, 2020; Witzke, 2009). (From a related mathematics education that explicitly takes this concept into account, we speak of so-called empirical-oriented mathematics classes, see Pielsticker, 2020). Our claim is not that students can formulate these theories, but that students behave as if they had these (student) theories. We want to understand an empirical theory to be a theory that describes and explains phenomena of reality (Balzer, 1982; Sneed, 1971; Stegmüller, 1986). The theories can have different sizes or ranges, so that everyday theories of children are also included (Struve, 1990).

In this context, the approach of empirical theories to describe student knowledge was explicitly developed in the work of Struve (1990) on a textbook analysis (textbook gamma) of geometry chapters. Empirical theories are thus in particular suitable to describe student knowledge in geometric contexts. An advantage of the theoretical approach is that it can be applied to situations where students perform actions on manipulative (according to the approach: empirical objects). In the structuralist presentation of an empirical theory–especially an empirical student theory–the terms (technical terms) are central shown in **Table 1**.

Table 1. Technical terms to describe empirical (students) theories in empirical contexts

Technical terms	Description
Empirical (student) theories	An empirical theory is a theory which describes and explains phenomena of reality (for the following as well, Sneed (1971), Stegmüller (1986), and Balzer (1982)). Empirical theories include all (natural) scientific theories, such as Newtonian mechanics and Maxwell's electrodynamics, but also economic or psychological theories, for instance. These theories can be of different sizes or scope, so that everyday theories of children are included as well (Struve, 1990).
Intended applications	Each empirical theory aims to describe and explain certain phenomena of reality. These phenomena are called intended applications. The intended applications of an empirical theory (corresponding technical term) are characterized exemplarily, i.e., they are defined trough the indication of paradigmatic examples.
Empirical objects	In this article, empirical objects are understood as items and objects of reality that are immediately accessible to students, especially in a tactile or visual way.
Empirical objects of reference	Empirical objects of reference are empirical objects which are covered by a concept. Furthermore, this is a term that refers to those objects which are considered to be definitory for empirical concepts in the sense of an empirical theory. As an example, the dice produced by students are objects of reference for the term "cube" used in the classroom
Non-theoretical term	T-non-theoretical terms and empirical concepts are terms of a theory T, which have been defined before the theory T was established (Burscheid & Struve, 2020). These include the "empirical concepts", i.e. terms that have empirical objects as objects of reference such as the term "triangle". For all theories T it is true that concepts describing objects of reality, or in other words having objects of reference in reality, are T-non-theoretical.

We take use of these technical terms to describe empirical (student) theories. In the spirit of the empirical theories approach, we assume that the students in our geometry case study experimentally validate and subsequently explain their findings from the mathematical learning situation. Knowledge explanation is about being able to explain gained knowledge independently and leaving behind the status of the "pure empiricist" and in this way being enabled to develop a conception appropriate to mathematics (Witzke, 2009). Thus, knowledge explanation gains a central importance for the development of mathematical (student) knowledge, because then real phenomena become explainable only in context, that is, only when we can trace them back to known statements (Witzke, 2009). In our case study, we want to complement this aspect of knowledge explanation by Fischer and Malle's (1985) concept of the "basis of argumentation". It is interesting to know when the students in our case study can conclusively trace their knowledge back to already known knowledge – a knowledge base – to be able to support their arguments.

What happens in school is the development of practical mathematics (Tall, 2013) and the formulation of theoretical definitions and deductions based on natural perception and reflection. In geometry the development as well as the production of empirical objects (manipulatives) in a variety of mathematical contexts seems quite easy and thus may facilitate empirical (or in Tall's (2013) terminology, "embodied") approaches to mathematical content (in accordance with our theoretical framework *empirical objects*).

Thereby we assume that students strongly tie their knowledge to the (empirical) context of its development (Bauersfeld, 1988). Research results show that a transfer of this knowledge, bounded to the contexts of its constructional process ("domain specific knowledge"), is a big challenge for mathematical learning in the sense of empirical theories (Burscheid & Struve, 2020).

To describe the students' developed mathematical knowledge of the concept of triangle in three empirical contexts we use the approach of empirical theories established by Burscheid and Struve (2020). These empirical theories provide a very clear analytical descriptive framework that explicates the aspect of knowledge development when dealing with empirical contexts.

METHODOLOGICAL DECISIONS

Type of Research and Population

With the aim of describing what knowledge students activate about triangles in different contexts with different (digital) tools and what content-related meaning do they give to the concept of triangles? we look at an interview situation with two girls Anna and Jules. We first look at the situation context of the students in our case study. In order to be able to describe a knowledge development process in empirical contexts, we first start with an account of the paper-and-pencil context, then we move on to the DGS (in the following "GeoGebra") context, and finally to the 3D pen context. All three contexts are empirical contexts, since empirical objects are used for the development of meaning to the triangle concept in all three contexts. In doing so, the three contexts were deliberately chosen for our case study. First, a context that is already common and well known to the students was made available. Paper-and-pencil is well known from school mathematics lessons and the students are familiar with this context. This provided the students with a good introduction to the interview situation. Subsequently, a digital/virtual empirical context should be in the foreground for the elaborations on the triangle concept. GeoGebra as DGS was chosen because this program is also used in mathematics lessons. Lastly, a context should then be chosen that probably does not seem so familiar to the two students. In addition, a digital tool should be used, which at the same time focuses on drawing. 3D pens allow for easy operation while at the same time drawing by hand.

Data Collection

The data collection was carried out by videotaping the learning environment based on the usage of 3D pen, GeoGebra, and paper-and-pencil in the context of the development of the concept of triangles. This is an interview of 1 1/2 hours in which the two students in our case study worked on the triangle concept. The data was collected in December 2021. The two students sat at a table in a separate room. The interviewer sat at a distance to the right of the two students. The student Anna (name changed) is seven years old and has already gained experience with triangles in everyday school life. The student Jules (name changed) is eight years old and also already has experience with triangles and the construction of triangles. These are students from a German elementary school. The two students have been noticed that they work particularly well together. At the same time, they are also able to endure conflicts and disagree. This made the two students stand out for our case study and the related research question. In addition, in terms of our research question, we wanted to select students who had only gained initial experience with triangles.

Data Analysis

In the spirit of an instrumental case study according to Stake (1995), the focus was on our research question: *What knowledge do students activate about triangles in different contexts with different (digital) tools (paper-and-pencil, 3D pen, and GeoGebra)? What content-related meaning do they give to the concept of triangles?*

The interview situation with Anna and Jules were transcribed according to the rules of Meyer (2010) and then analyzed using the approach of empirical theories (**Table 1**). Key scenes are selected with a view to generating informativeness or a deeper understanding of the case ("case studies are undertaken to make the case understandable", Stake, 1995, p. 85). We wanted to find out why (Taber, 2014). The two students and their concept formation processes of triangles in empirical contexts therefore a specific case about which we wish to know about. "We need to learn about that particular case" (Stake, 1995, p. 3). As is usual for an instrumental case study, we are interested in a particular case in order to investigate an issue and gain knowledge. We want to uncover knowledge about the phenomena of interest (Stake, 1995). An important contribution to the contextualization of our case study is provided by the interview with the two students.

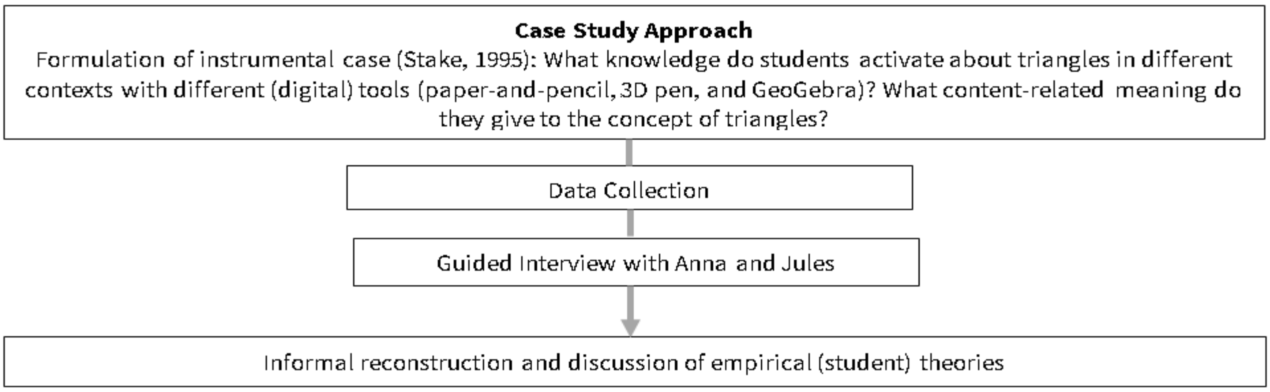


Figure 1. Summary of the methodological decisions

Table 2. Transcript from the interview: What is a triangle?

I	9:55	What is a triangle? And try an answer without drawing.
A	10:01	I already know what a triangle looks like (Takes her pencil and draws a triangle in the air). 
[…]		
J	11:00	Maybe a triangle does not always have to look like this (draw a triangle in the air in front of you with your finger), but only needs three corners.
I	11:05	Aha. Okay, three corners, we'll hold on to that for now. (4s) What else do you remember or what else do you think about a triangle?
A	11:10	And that it must always have three sides. Not four sides, for example.
I	11:15	Okay, so we have established?
A	11:17	That a triangle must have three corners. And three sides.

For the analysis of the specifics in the student concept formation processes while dealing with the three contexts we use a descriptive perspective. Therefore, we make use of the technical terms of **Table 1**.

Figure 1 shows how we have advanced in our methodological framework. At first, we formulated the research question, which define the case study. In the next step we gathered data in controlled clinical interview situations. Therefore, we selected a pair of students who know each other well but are of heterogenic mathematical background. The recordings were transcribed and form the basis of our reconstructive research analysis. This has the aim to describe activation processes of mathematical knowledge in local (student) theories. The interpretation is led by a constructivist research paradigm namely that individuals develop their knowledge in interaction with real phenomena.

ANALYSIS AND RESULTS – CASE STUDY IN EMPIRICAL CONTEXTS

Triangle in Paper-and-Pencil

Anna and Jules are asked at the beginning of the interview what you understand by a triangle. First, Anna answers as shown in the following transcript. Both students are asked to answer first without using a drawing or similar. It should be noted that the transcripts for this paper were translated from German into English.

As can be seen in the transcript shown in **Table 2**, Anna almost automatically uses an object to clarify her triangle concept. She immediately uses a pencil and, in a sense, "draws" a triangle in the air. She focusses on gestures to give meaning to the concept of triangles. She probably takes into account that no triangle should be drawn on paper but clarifying her triangle concept in the air in front of herself seems unproblematic from her point of view. Anna thus immediately uses an empirical object to "show" the meaning of the triangle concept. Jules then provides a stimulus and focuses on characteristics such as "needs three corners" (**Table 2**, 11:00). Jules focuses on empirical properties that she believes are characteristic of a triangle. Anna can subsequently add: "it must always have three sides" (**Table 2**, 11:10) and takes over Jules' impulse. The common answer "needs three corners and it must always have three sides" of the two students, we want to describe as the basis of argumentation (Fischer & Malle, 1985). Anna and Jules come back to this basis of argumentation again and again in the course of the case study.

Figure 2. Anna's triangle

Table 3. Transcript from the interview: Properties of triangles

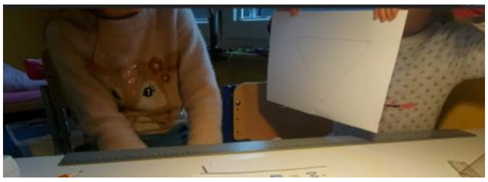
I	13:50	Why is this a triangle now?	
A	13:53	Because it has three corners and three sides (Mark each corner with the pencil as if Anna wanted to draw an angle in it). First corner, second corner, and third corner. Three corners. (Anna holds the piece of paper up in the air). Oh the wrong way around (She turns the paper over).	 

Table 4. Transcript from the interview: Discussion on turning the paper with the triangle drawing

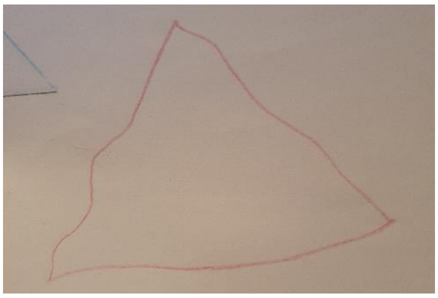
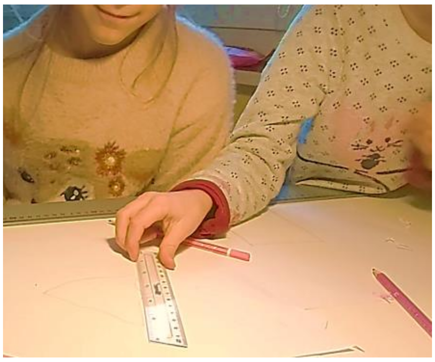
I	14:21	Is that how you want to hold it?
A	14:24	Yes.
J	14:25	Does not it has to look the same from all sides?
I	14:27	Does it have to?
A	14:28	It is (and holds the paper up in the air again and now turns it several times). Like this, like this, and like this.
J	14:30	Because, all sides have to be the same length. I think so, anyway.
A	14:34	No. (looks directly at Jules). A triangle can have equal sides and not have equal sides. So that's how a triangle is. (Holds the triangle up in the air again as before, image from **Table 3**, 13:53, II).
I	14:53	What if you now hold the triangle like this (Anna turn the paper in that way that one triangle corner reaches down towards ground).
J	14:59	Still has three corners. But it is crooked. I do not know.
A	15:05	But it's not supposed to have four corners. (4s) But it's just such a triangle (holds the sheet of paper again as in **Table 3**, 13:53, II). However it's a triangle the wrong way around.
J	15:25	So, I guess it's a triangle from all sides.
A	15:30	Not from this (Points to the side where one corner of the triangle is facing down).
		(Anna and Jules discuss incomprehensibly)
A	16:25	It's not a triangle from all sides. Because it's not a triangle like that (Holds the sheet of paper again so that one corner of the triangle points downward).
J	16:30	But it still has three corners.

Subsequently, paper-and-pencil is to be worked in the empirical context. For this purpose, the two students of our case study are asked by the interviewer to draw a triangle on the paper. Again, Anna begins. Anna takes a ruler, paper and pencil and starts drawing her triangle. Anna draws a triangle (**Figure 2**) with one side of the triangle parallel to the edge of the leaf (see also Pielsticker, 2021).

After Anna is finished with her drawing, the interview asks again, "why is this a triangle now?" Anna then holds the triangle in front of her so that the interviewer can see her drawing. She notices that she thinks she is holding the triangle the wrong way around and turns the sheet with the drawing around. This becomes clear in the transcript excerpt shown in **Table 3**.

At this point, it is interesting that Anna rotates the paper with the triangle. She wants one side of the triangle to be parallel to the ground, not a triangle corner. In the next transcript shown in **Table 4**, the two students discuss why Anna turned the paper and what impact this may have on the concept of triangle.

Table 5. Transcript from the interview: A triangle with waves

I	34:35	I'm going to draw a triangle now (draw a crooked triangle).	
A	35:10	That has quite a lot of waves. Is it still a triangle? Yes, it is still a triangle, because it has three corners. It's just a little wavy. And one side is longer than the other. One side has waves and the other doesn't.	
J	35:34	(Points to a wave of a triangular side). Is this a wave or not?	
A	35:40	That's a corner, because that's pointed (points his finger at the shaft and forms a point). So, it's not a triangle. Then it is a square.	
I	35:50	I think the question is quite good Jules. Would you see that as a corner or not?	
J	36:02	It is not as pointed as this one (points to a triangle corner), so it can't really be a corner. But rounded.	
I	36:32	Anna, what would you say. Are there rounded corners.	
A	36:50	There are no rounded corners in the side. So, there are no corners in the side.	
J	37:02	I suspect it's not a triangle.	
I	37:12	Why?	
J	37:18	It has three corners, like the other one. But in the triangles, which one knows, is no round thing in it. Such a rounding.	
I	37:42	So, what is the problem with this triangle?	
A	38:20	The problem with the triangle is that it has not been drawn straight (10 sec.) So, it is not drawn neatly. With the ruler (puts a ruler next to a triangle side with waves). Here you have a complete curve in it.	

The two students in our case study hold different opinions at this point. Anna wants the triangle to be seen with a triangle side down and not with a triangle corner. Anna assumes that it is a triangle because the properties discussed earlier (**Table 2**) are preserved. "Still has three corners. But it is crooked" (**Table 4**, 14:59). If the triangle is held in a different way, she refers to it as "crooked". At the same time, Jules indicates that it must be a triangle from all sides (**Table 4**, 15:25). Then, in 15:35 (**Table 4**), Anna is no longer sure if it is really still a triangle when the drawing is rotated on the drawing sheet. The two students then discuss in whispers, which remains incomprehensible to the interviewer. In the end, however, the two cannot overcome their conflict. Even though the properties of a triangle remain the same for Jules and for Anna, Anna argues that the "crooked" and rotated triangle in the drawing sheet (with one triangle corner parallel to the ground) is not a triangle "from all sides" (**Table 4**, 16:25).

Then the interviewer draws a triangle himself. At Anna's request, the color pink is chosen. The interviewer draws one of the sides of the triangle wavy. According to a Euclidean understanding of geometry, the figure drawn by the interviewer is not a triangle, since according to axioms set up there, a line is the shortest connection between two points. The drawn figure leads to the discussion shown in **Table 5**.

In the transcript excerpt (**Table 5**), it is clear how the two students struggle over the meaning of the triangle concept. Anna initially sticks to the definition of three corners and three sides that they established earlier. "That has quite a lot of waves. Is it still a triangle? Yes, it is still a triangle, because it has three corners" (**Table 5**, 35:10). At this point Anna refers to the basis of

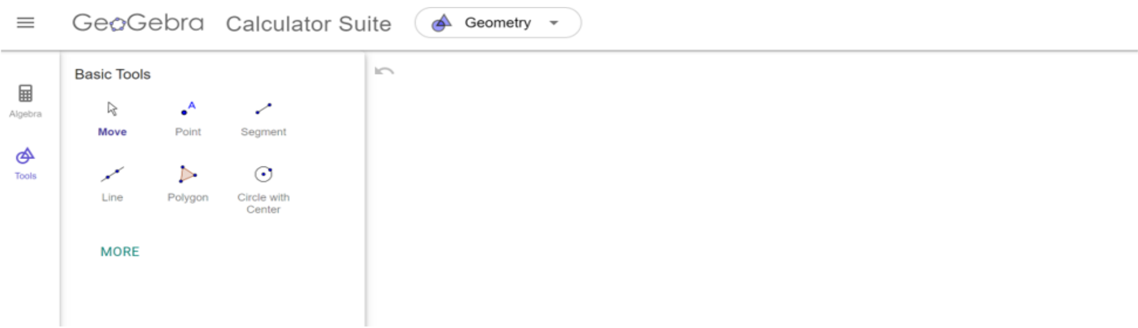


Figure 3. "Geometry" platform in GeoGebra

argumentation - "needs three corners and it must always have three sides". For Anna, the triangle is "just a little wavy" (**Table 5**, 35:10). Afterwards, the two students discuss whether the "wave" in the side of the triangle is not or cannot be seen as a "triangle corner". Anna then states that it would then be four corners and thus the figure would not be a triangle, but a quadrilateral (**Table 5**, 35:40). Jules states that it cannot be a corner, it would be more something "rounded" (**Table 5**, 36:02). Anna then states: "There are no rounded corners in the side. So, there are no corners in the side" (**Table 5**, 36:50). After this comment by Anna, Jules argues that the figure is not a triangle. The student justifies this with the fact that none of the triangles she knows has such a rounding or wave (**Table 5**, 37:18). An empirical reference becomes particularly clear in this justification of the two students. Jules brings an ostensive explanation (Struve, 1990) at this point. Jules refers for her reasoning to other examples of triangles she knows well. And these known examples of triangles have no rounding.

The students are thus bothered by the "wave" or the rounding in the side of the interviewer's drawing. Anna even puts the ruler next to the side of the triangle with the wave as a further argument that it was not drawn "straight" (**Table 5**, 38:20). The two students thus argue on the empirical object–on the interviewer's drawing page figure. It is interesting to note that the drawn triangle fulfils the student criteria of three corners and three sides (basis of argumentation). Nevertheless, the two students deviate from their established criteria or develop them further with the help of the empirical objects of reference. For them, a triangle (as a non-theoretical term) should also be drawn "straight" and not contain any curves or waves (Euclidean geometry understanding). Anna supports her argument, as can be seen in the last figure (**Table 5**), also with an aid, the ruler. Anna brings another argument into the discussion. Anna brings an operational explanation (Struve, 1990) to the discussion about the "wave" in the side of the triangle. This becomes clear by the fact that she uses the ruler as an illustration of her explanation and places it at the wavy side of the triangle. Anna justifies with the help of the ruler and a possible construction of a triangle. At this point, however, it would also be possible for Anna to describe a more naive-empirical understanding of the concept of a triangle. In the sense that Anna, using the ruler, only wants to make clear that the side of the triangle in question "does not look good" and is not "straight".

The students develop their empirical student theory about triangles on the drawing sheet on the empirical object (further).

In the following situation, the two students in our case study are working with GeoGebra. Again, Anna and Jules are asked to construct a triangle in the program and then describe it.

Triangle in GeoGebra

Students construct a triangle in GeoGebra. The "geometry" environment of GeoGebra was used for this purpose. Here the students had the option to "move", "point," or "segment" and others (**Figure 3**).

The laptop is placed in front of Anna and Jules. The students work together on a laptop. Anna wants to start immediately with a construction when the interviewer first asks whether it is possible for the two students to draw a line. When this works without any problems, the interviewer asks whether Anna and Jules can also construct a triangle. Thereupon Anna starts with her triangle construction. The process is clarified in the following transcript excerpt (**Table 6**).

Anna immediately begins with a construction. She selects "segment" in GeoGebra and starts with a first stretch. In between she still has to make changes. Jules does not seem to be satisfied with Anna's construction. In her opinion Anna's construction has too many corners. Again, she uses the criterion set before by the two students that a triangle must have three corners: Basis of argumentation–"needs three corners and it must always have three sides". Anna is not quite clear at first what Jules means and asks, "where are corners now?" (**Table 6**, 52:21). As can be seen in Anna's construction, her construction includes first six "points" and then seven "points". This gives hints that she does not consider these "points" as corners in the triangle, unlike Jules. For Jules, the "points" are corners of the triangle. It seems that Anna also expects similar corners in the GeoGebra program as in the paper-and-pencil context. The two students continue to discuss (**Table 7**).

As seen in **Table 7**, 52:25, Jules sees the "points" as corners and tells Anna that her construction created too many "points". Jules sees the reason for this in the fact that Anne has always started anew in the program or has started anew for a distance. This procedure is not problematic in the context of paper-and-pencil. Anne's goal was the construction of a triangle. At first, the operating elements (e.g. points, segments) of GeoGebra do not seem to be decisive for her. It seems that Anna wants to transfer her context of paper-and-pencil and the empirical experiences in this context to GeoGebra. Thus, she tries to draw in a similar way as she would do on the drawing sheet. "Interruptions" would not be an obstacle on the drawing sheet. Anna also does not pay attention to the "points" for her construction. She focuses on the three sides that are to be created. In some ways, Anna "simply" reads the representation of a triangle created in GeoGebra differently than Jules. Jules seems to be more clearly tied to the empirical object in the GeoGebra context. For her, the "points" are the corners of the triangle construction. Moreover, the triangle

Table 6. Transcript from the interview: Triangle in GeoGebra–Anna’s triangle construction

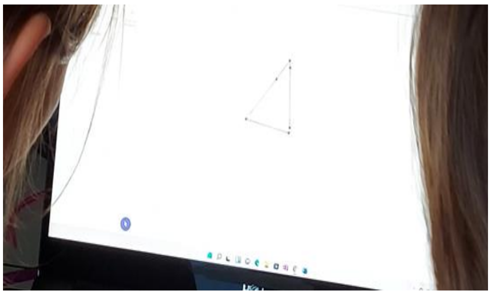
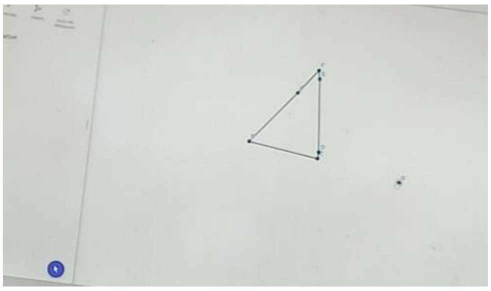
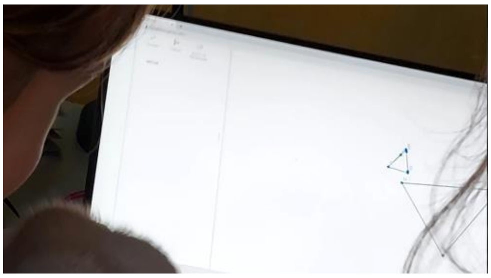
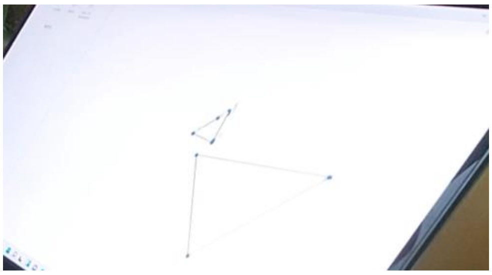
I	50:26	Can you draw a line?
A	50:32	(Draws a line with GeoGebra).
I	50:54	Great. Can you also draw a triangle?
A	51:56	Yes. (Sets a point in the program, draws the line, sets a point again, etc.). This is a triangle. That is a triangle (Makes another change).  
J	52:10	(Shakes her head). These are more corners.
A	52:21	Where are corners now?

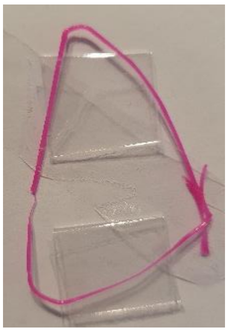
Table 7. Transcript from the interview: Triangle in GeoGebra – to many corners

J	52:25	Here you started over and further up there. Not further straight up. If you went from C to D and then from D to E, you made a corner at D. And from E to F you made a corner at E again.
I	53:32	And how many corners do we actually need?
J	53:39	Three.
I	53:43	Do you want to try (to Jules) to draw a triangle?
J	53:50	(Draws). So if you draw each corner you can make a triangle out of it again.  
I	57:36	I see. How is the triangle in the program different or the same from the previous triangle on the drawing sheet?
A	58:22	This is painted on the sheet and this is not (points on the screen). But it’s actually the same if you turn it like this (turns the paper around with the previous triangle).
I	58:45	What are you saying Jules?
J	58:50	(Takes sheet of paper with previous triangle in hand). That looks different?
I	59:05	Why?
J	59:10	Because here (pointing to the paper) we made the corner all pointy and here (pointing to the screen) they are not made pointy.

should be “drawn in one”. For this purpose, she starts with the triangle corners (points) and then adds the triangle sides (segments): “if you draw each corner you can make a triangle out of it again” (**Table 7**, 53:50).

The answers of the two students to the interviewer’s question: How is the triangle in the program different or the same from the previous triangle on the drawing sheet? (**Table 7**, 57:36) are also interesting.

Table 8. Transcript from the interview: Triangle with 3D pen

I	01:08:02	Creates a triangle with the 3D pen.	
	01:08:10	(The two students start the construction process).	
I	01:10:23	What properties does your constructed triangle have now? You can also compare it with the other ones.	
J	01:10:30	Hmm, did not become completely straight. Here it is so double and here too.	
I	01:11:04	Is this even a triangle now?	
A	01:11:10	Not a triangle. Because there it is double and kind of a three corner.	
J	01:11:25	And here is another corner.	

Anna answers first. For Anna, the only thing that matters are how the triangle is turned on the drawing sheet: "it's actually the same if you turn it like this" (**Table 7**, 58:22)–an ostensive argument (Struve, 1990). Anna extends her student theory about triangles on drawing sheet figures to include triangle construction in GeoGebra. For Jules, the two triangles, on the drawing sheet and in GeoGebra are different (**Table 7**, 58:50). It is interesting that she justifies this again with the corners. In the program, the corners look different to her than on the drawing sheet. On the drawing sheet the corners are "pointy"–again an ostensive argument (Struve, 1990).

In this context, students again work with empirical objects. It is interesting that Jules pays special attention to the specifics of the GeoGebra program, whereas in Anna's empirical student theory, the controls make no difference. Anna uses "points" or "segment" for her construction, but these have, for example, no relevance for the criterion of three corners. She does not connect the controls of GeoGebra with the criteria. She uses GeoGebra similar to what she is used to with paper-and-pencil.

The students then work with a 3D pen. Again, a triangle is to be constructed in this context.

Triangle with 3D Pen

In the following transcript, a triangle will be created using a 3D pen. For this purpose, the two students are only now handed two 3D pens. Each student has her own 3D pen and can work in parallel. As is clear in the transcript below, the two students are asked to construct a triangle using the 3D pen. Once the 3D pens were heated up, they both begin their triangle construction. They use a blank drawing sheet as a base for their drawing (**Table 8**).

In their construction with the 3D pen, both students pay special attention to the corners of the triangle drawing. As the transcript shows, there are too many corners in the triangle construction. Probably caused by the threads of the 3D pen, there are too many corners, which are criticized by the students. This bothers both students (**Table 8**, 01:11:10 & 01:11:25). At this point, the two students seem to apply their criterion of the three corners in the triangle (basis of argumentation). They evaluate the emerged empirical objects–3D pen triangle. In the process, the created threads seem to interfere. Interesting is also the following discussion about whether the same triangle can be drawn again. The transcript excerpt, shown in **Table 9**, is intended to illustrate this.

Student Jules' rationale in response to the interviewer's question is interesting. With the 3D pen, it does not seem possible to Jules to draw the empirical object of the triangle again in the same way. She justifies this by the action to be performed: "I don't move my hand the same way every time" (**Table 9**, 01:12:00). This makes clear that actions play a special role for the student in the context of the 3D pen. The argumentation with drawing actions seem to be more prominent here than in the paper-and-pencil or GeoGebra context. The empirical object of the triangle with 3D pen seems to have been created for Jules in particular by the

Table 9. Transcript from the interview: The same triangle

I	01:11:30	Can you draw a triangle that looks the same?
J	01:11:43	I think that will be rather difficult.
I	01:11:51	Why?
J	01:12:00	Yes, I don't move my hand the same way every time.

plot. The action with one's own hand seems to become more obvious using the 3D pen than paper-and-pencil, where work is also done with the hands. This is surprising at first. But if we go back to the paper-and-pencil context, it becomes more obvious that Jules cared about accurate drawing. And this was achieved by the ruler. With the 3D pen, exact drawing is difficult. In addition, the feeling arises that the (wrong) movement with the hand with the 3D pen immediately has a large effect on the resulting triangle. It may also seem more difficult to the students to improve a mistake in the drawing process. In a construction with paper-and-pencil, an error in the drawing could be improved more easily with pencil and eraser. Thus, in the 3D pen context, the empirical object and the actions performed on it seem to be very clearly in the foreground.

CONCLUSIONS

At this point, we would like to summarize the three empirical contexts and the described development of students' knowledge of the triangle concept. To do this, we specifically address our research question:

What knowledge do students activate about triangles in different contexts with different (digital) tools (paper-and-pencil, 3D pen, and GeoGebra)? What content-related meaning do they give to the concept of triangles?

When Anna is asked about her triangle concept in the paper-and-pencil context, she immediately seems to think of an empirical objects of reference. She makes this clear by drawing a triangle in the air in front of her body with the pencil. In particular, an object-bound triangular concept becomes nicely clear from this. The student Jules focuses on the characteristic properties of a triangle: three corners. She focuses on empirical properties of a triangle that can be described. Anna can pick up Jules' level of description and adds the property of three sides. At the same time, it seems important to both students that the triangle is not "crooked" and is also drawn "straight". This is especially noticeable when the two students discuss the interviewer's triangle. The interviewer's drawn triangle contains a wave in one of the triangle's sides (according to a Euclidean understanding of geometry, the figure drawn by the interviewer is not a triangle, since according to axioms set up there, a line is the shortest connection between two points). Here, Anna and Jules seem to have an empirical object of reference–triangle–to which they refer in their student theory. This triangle is drawn on paper and has three corners, three sides (basis of argumentation), is not "crooked" or has a wave in it.

In the GeoGebra context, it is interesting that Anna's "points" are not considered or perceived as corners of a figure. In GeoGebra, it is important to her that the triangular shape becomes recognizable and focuses on the triangular sides for the development of the meaning of the triangle in this context. In contrast to Jules, Anna is not bothered by the "points" in the program that are labeled with letters. Anna seems to get vague about her basis of argumentation in this GeoGebra context. She looks over the "points" in a certain respect. Anna seems to allow a degree of inaccuracy at this point. In the GeoGebra program context, the "points" are supposed to represent corners. Anna could not draw the triangle any better with the program and may say to herself kind of "it fits". Nevertheless, due to the complains of the program Anna draws a polygonal train and not a triangle. Thus, the student Jules has an argument, since Anna's figure in GeoGebra is not a triangle. Anna allows a certain degree of inaccuracy–"close enough", unlike Jules. Jules uses the argument base of the beginning. In **Table 6**, Jules states that there are too many corners. A deductive argument. Jules does not hold that the triangle is crooked, or a polygonal train. Jules uses the argument that triangles always have only three corners. She sticks to the basis of the argument–an empirical argument from the students' initial situation (see **Table 2**). We can thus state: Anna transfers essential elements of her empirical theory about drawing figures into the program. Thus, for example, that setting down the pencil (interruption in GeoGebra) does not create a corner. Thus, the triangle remains. Jules, on the other hand, interprets the interruption in GeoGebra as a reattachment, which creates new corners for you. Thus, no triangle is created for her. Jules uses the argument "in a tracing" at this point. Both students argue differently based on their empirical perception and interpretation of it.

A particular focus on actions on empirical objects seems to emerge in the 3D pen context. This is particularly evident in Jules' statement, "I don't move my hand the same way every time" (**Table 9**, 01:12:00). At the same time, at this point the drawn straight triangle again becomes clear as an object of reference for the students. The triangular figure, which was created by the 3D pen, is not exact and straight enough for the two students. Here again a reference to an empirical object of reference of the two students becomes clear.

The intended application of the context paper-and-pencil can be described as drawing a triangle with the help of paper, pencil and ruler. The intended application of the GeoGebra context can be described as creating a triangular figure using the "points" and "segment" operating elements. And as an intended application in the 3D pen context, the creation of a triangular figure can be described by liquefying PLA and guiding the 3D pen.

Different intended applications can be described for all three empirical contexts. Here, the two students in our case study use the three empirical contexts to develop the meaning of their triangle concept. Either it is about finding a contrast, or about finding common characteristics. According to Fischbein (1993), students express their knowledge about triangles referring to ideal prototypical forms in their mind. With Fischbein (1993), we would have come to considerably different interpretations. For him there is something like an ideal knowledge of abstract entities detached from realistic drawings.

> “The geometrical figure itself is only the corresponding idea that is the abstract, idealized, purified figural entity, strictly determined by its definition” (Fischbein, 1993, p. 149).

With this research paradigm in mind, one would have to conclude that the knowledge is not transferred from one context to another but only shows itself in different contexts. At the end, from an epistemological point of view it is not decidable if this rather platonic or our rather constructivist approach is true. But they do lead to considerably different interpretation. It is up to the reader to decide if in compliance with his/her own belief system Geometry is a science about ideal perfect platonic bodies or a science about imperfect physical objects in our environment. If the reader is interested of a deeper discussion of that we suggest reading the article “geometry and empirical science” of Hempel (1945), where he distinguished between pure and physical geometry.

Anna and Jules in accordance to our theoretical framework clearly show an empirical understanding of triangles. They argue based on the empirical perception (empirical properties in the contexts) and interpretation of it. According to Polya (1945), geometry can be seen as the right reasoning on wrong figures. The two students take the preciseness of the mathematical constructions into account (this reminds of Schoenfeld’s (1985) concept of pure empiricism). Unsurprisingly, this understanding is clearly grounded in empiricism. The knowledge has obviously been developed on the basis of empirical characteristics of the figures and the concepts develop in the context of practical mathematics. With regard to our initial quotation from Hefendehl-Hebeker (2016), Anna’s and Jules’s behavior appears to be as perfectly sound as an authentic mathematical activity.

Under these premises, it seems necessary that mathematics education has viable description possibilities for dealing with empirical contexts. Therefore, further empirical contexts (especially with a focus on digital media and tools) should be considered. Specifically, it could be interesting for our study to provide the students with a definition of a triangle construction. Also, it would be exciting to observe how Anna’s and Jules’ reasoning would change if, for example, the focus is on the triangular area. However, careful consideration must be given to what an empirical context can do (what actions are feasible) and what is not. In this context, it can be assumed for our study that the students bound their triangle concept to an empirical reference object. Thus, for an outlook on our study, we can link to Bauersfeld’s (1985) 4th thesis, which is that there are no general concepts, strategies, and procedures. The subject can think them generally, but they are not generally available, i.e., they cannot be activated independently of domains. The question we then have to face (also with reference to Bauersfeld, 1985) is: Can a concept of triangle detached from the empirical context be developed or thought of and is it also desirable for mathematics education?

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