International Electronic Journal of Mathematics Education
2024, 19(2), em0774
e-ISSN: 1306-3030
https://www.iejme.com

Research Article

An exploratory study of spontaneous representations of covariational reasoning in middle school students

Ulises García-Teutli [1*], José Antonio Juárez-López [1]

[1] Benemérita Universidad Autónoma de Puebla, Puebla, MEXICO
***Corresponding Author:** ulises.garciateu@alumno.buap.mx

Citation: García-Teutli, U., & Juárez-López, J. A. (2024). An exploratory study of spontaneous representations of covariational reasoning in middle school students. *International Electronic Journal of Mathematics Education, 19*(2), em0774. https://doi.org/10.29333/iejme/14386

ARTICLE INFO

Received: 23 Nov. 2023

Accepted: 09 Mar. 2024

ABSTRACT

Although student covariation reasoning has been explored in depth to improve understanding of the correspondence between variables, research has focused on studying existing reasoning about variables in Cartesian representations. The working method had a qualitative approach, with a descriptive exploratory scope, the spontaneous representations that the participants evidenced under the level of covariational reasoning of the variables present in three contextualized situations were explored, posed to a population of third-grade middle school students. The students argued with concrete and abstract drawings the general behavior of the variables. At low levels of reasoning they used pictorial representations, at higher levels they used graphical diagrams and tables. The exploratory study shows a relationship between the type of spontaneous representations and the student's level of covariational reasoning, as well as the rigor of the description of the problem.

Keywords: covariation, spontaneous representation, exploratory study, covariational reasoning, middle school students

INTRODUCTION

The development of mathematical reasoning in students is an essential process in the integral formation and strengthening of their cognitive skills. As students advance in their education, mathematical reasoning becomes a necessity to tackle complex problems, understand abstract concepts, and apply mathematics in real-life situations. Copi (1978) argues that "all reasoning is thinking, but not all thinking is reasoning" (p. 18). Thinking is considered as that mental discourse, a mental reflection of the person that allows him/her to create ideas and concepts (Copi, 1978; Polya, 1984; Vygotsky et al., 1995). On the other hand, reasoning is a special process of thinking that arises from premises and conclusions of an absolute truth are reached (Copi, 1978; Peirce, 1903 cited in Campos, 2010; Syarifuddin et al., 2020).

Each person manifests a different reasoning derived from the work method used to obtain conclusions. From these differences, three main types of reasoning are identified: sequential, inductive and quantitative (Carroll, 1993). Through quantitative reasoning, one can start from the facts present in tasks that involve mathematical or quantitative relations of concepts to arrive at the truth (Martínez-Miraval & García-Rodríguez, 2023; Thompson, 1993). With quantitative relationships, Thompson and Carlson (2017) describe that in mathematical reasoning a situation is conceptualized in terms of quantities and relationships between quantities, strengthening this reasoning in students will result in improving the creation, manipulation and interaction between variables created to represent situations.

In Thompson's (1993) theory of quantitative reasoning, variation and covariation make sense to explain how students manage to interpret a situation quantitatively, while at the same time they imagine how events change with the values of quantities that are varying.

A quantity is conceived as the quality of some aspect of a situation that the person considers as accounting, it may have a numerical value, but this data is not required for reasoning processes (Kaput, 1995; Thompson, 1993). In an accounting situation, the presence of objects that generate the need for measurement is obviated. Consequently, the appearance of a single value and separated from reality loses meaning, more than one value is required to compare such measurement of values in a quantity, this is evidence of the change in an object. For Thompson (1993), the change in the value of quantities can have different meanings depending on the interpretation of the person. If the value never varies, the symbol representing that value acquires the meaning of constant. If that value changes from one environment to another, but not in the same environment, it acquires the meaning of parameter. If it changes within the same environment, the symbol has the meaning of a variable. Variation and covariation are presented for Thompson (1993) as the way to explain students' reasoning when conceptualizing a situation (Thompson & Carlson, 2017). In this sense, "a person reasons covariationally when he imagines that the values of two quantities vary and visualizes them varying simultaneously" (p. 425).

When students are confronted with situations involving the analysis of quantities and change, the level of covariational reasoning exercised largely determines their ability to answer correctly and to understand how the situation develops. At the secondary level, a need arises to process elements previously static for the student (numbers, figures, etc.) as dynamic objects that respond to algebraic tools and that are present in situations involving functional relationships. There are registers of the difficulties that arise in students when these reasoning processes do not reach the expected scope, as in the learning of functions (Bagossi et al., 2022; Rolfes et al., 2021; Thompson & Carlson, 2017). These difficulties, research is directed in principle to the study of covariation and the student's level of reasoning, in pre-class stages on the elaboration of planning or with the analysis of introductory tasks and instructions in basic level (primary and secondary), even with the instructional strategies that the teacher uses to facilitate reasoning, in post-class stages with the analysis of the construction of mental images obtained from dynamic situations and the ratio between covariate quantities (Castillo-Garsow, 2012; Johnson, 2015a; Martínez-Miraval et al., 2023; Miranda & Sánchez, 2019; Oehrtman et al., 2008; Panorkou & Maloney, 2016; Passaro, 2009; Swidan et al., 2022).

Other works analyze covariation with theories that delve into the type of problems or methodological frameworks that address covariational situations. Textbook exercises are analyzed, student mental processes when modeling situations, even connections between covariation and emotions of achievement when successfully understanding an activity are addressed (Bagossi et al., 2022; Carlson et al., 2002; Chen, 2023; Confrey & Smith, 1995; Nava-Guzmán et al., 2023; Syarifuddin et al., 2020; Tasova et al., 2020). Much research is qualitative in the analysis of covariational reasoning since each student conceives and represents dynamic events at different levels. Most of these works give priority to this kind of analysis when dealing with higher level concepts, such as limits, derivatives, and integrals (Bagossi, 2023; Carlson et al., 2001; Confrey & Smith, 1995; Martínez-Miraval et al., 2023; Mkhatshwa, 2023); other works approach, where covariation acquires a first rigor with the definition of variables, proportionality and change (Ellis et al., 2020; Hitt & González-Martín, 2015; Johnson, 2015b; Panorkou & Germia, 2021; Passaro, 2009; Yu, 2024).

The present work follows a similar line to the research on covariational reasoning in middle school students, who still train covariational reasoning with dynamic situations, although it is beneficial to focus on specific topics or the design and analysis of educational material. In the works it is common to find activities or items elaborated under the structure of the tasks of Carlson et al. (2002) or Thompson and Carlson (2017), where covariational reasoning is explored from a Cartesian graph that is requested to the student with the variables of the situation. Hence, Carlson (2002) distinguishes in the levels of covariational reasoning a mental action linked to coordinate axes, construction of secant lines, smooth curves, etc., as well as Thompson's (1993) integration of chunky and smooth continuous covariation into the levels of covariational reasoning. The result of the relationship between variable quantities that allude to the design of a function, subjected in practice to the possible design of Cartesian graphic tables of values.

The present interest resides in studying how the student reasons covariationally and interprets solutions, that is, to identify which elements he works with and knows how to use to correlate the variables in a situation without requesting a Cartesian graph or a table of values, allowing him to freely express his understanding of the problem. Even though the level of covariational reasoning may be adequate to describe a situation, writing incomplete procedures or designing confusing representations may undermine the ability to interpret and communicate the real situation, together with the solution process.

Research Questions

From the research described above, it is possible to appreciate the need to explore covariational reasoning in students, mainly at academic levels, where the concept of variable is taught and where they begin to work on dynamic situations, since poor or mistaken reasoning can generate difficulties in understanding the situation presented. Considering the importance of identifying covariational reasoning in students, as well as the representations they use to solve a problem, the following questions are posed: What level of covariational reasoning does the student show when faced with situations that require the recognition of variables and the correspondence between them? What is the spontaneous representation that emerges in the student when interpreting the dynamic situation?

Justification

The working methodology of the students to face mathematical situations mostly changes according to the academic level, since the curriculum of each educational system, institution or academic program determines part of the knowledge that the students will be presented with. The student's understanding of the situation, the mathematical language he/she masters to transcribe the mental images created and to carry out the corresponding interpretation, adjustment and modeling of the resolution processes also have an influence. Even when the student can develop a level of covariational reasoning, the resolution process depends on the interpretation of the problem, which responds to the knowledge he/she has developed; therefore, it is important to identify which characteristics are present in the student's response to covariation situations and his/her reasoning processes.

The interpretations that the student generates are an essential factor in mathematics education, which is the reason for studies that have been devoted to identifying their equivalence with knowledge, meaning, understanding and modeling (Font et al., 2007). As Goldin and Janvier (1998, cited in Font et al., 2007) point out, in the teaching and learning of mathematics, a representation can be subject to an external situation or set of situations, a language system with semantic and syntactic structures, formal mathematical constructs and an internal cognitive configuration.

Table 1. Major levels of covariational reasoning (Thompson & Carlson, 2017)

Level	Description
Smooth continuous covariation	The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, & person envisions both variables varying smoothly & continuously.
Chunky continuous covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, & they envision both variables varying with chunky continuous variation.
Coordination of values	The person coordinates values of one variable (x) with values of another variable (y) with anticipation of creating a discrete collection of pairs (x, y).
Gross coordination of values	The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases". The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values.
Precoordination of values	The person envisions two variables' values varying, but asynchronously–one variable changes, then second variable changes, then first, & so on. The person does not anticipate creating pairs of values as multiplicative objects.
No coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.

THEORETICAL FRAMEWORK

Quantitative reasoning is the ability to analyze quantitative information of a situation, and the relationship between the quantities involved to use procedures with which it is possible to get to a solution (Dwyer et al., 2003; Thompson & Carlson, 2017). To face a situation requires a mathematical support that represents the objects described, quantifying the situation allows the student to develop the problem. According to Dwyer et al. (2003), there are six competencies that a student should have to develop quantitative reasoning: read and understand information in various formats, interpret and embody quantitative information, solve problems mathematically, estimate and revise answers, communicate quantitative information, recognize the limitations of the methods used. In this work, priority is given to the communication of the student's written information and the limitation of the method used to solve the problem.

Jere Confrey and Pat Thompson (1980-1990) constructed the theory of covariational reasoning that describes the coordination of values of two variables during a change phenomenon and the conceptualization of variable quantities followed by their simultaneous changes. These first ideas, separated by the research objectives of both authors, are unified in subsequent research, with this covariation is formally defined as: the cognitive activities involved in the coordination of two quantities supported under a framework in which they vary according to the forms in which one changes with respect to the other (Carlson et al., 2002; Saldanha & Thompson, 1998).

The images that the student constructs must make sense to correctly describe the development of the dynamic situation, since a student will not always use the definition to describe a mathematical object. Vinner and Dreyfus (2003), define the image as "the set of all the mental images associated in the student's mind with the name of the concept, together with all the properties that characterize it" (p. 356). They also describe that these mental images are a type of representation, such as images, symbols, diagrams, etc.

Carlson et al. (2002) describe the behaviors that are appreciated when students participated in covariation tasks. These behaviors are classified under mental actions that allow identifying the covariational reasoning ability of a particular task. A student's behavior is classified into one of five levels of mental action according to the global image exhibited by the student of the context of the problem, and the level of covariational reasoning is obtained when the student sustains the mental action associated with that level and the actions of the lower levels.

On the other hand, Castillo-Garsow (2010) focuses the construction of covariational reasoning in the framework of the student's variational reasoning, that is, the way in which he coordinates the values of a variable quantity separated from the other variables (before the correlation between quantities). Thompson and Carlson (2017) concentrate these ideas in a table of variational reasoning, matching the five levels of the mental actions of Carlson et al. (2002) but with study on a quantity and the reasoning of the change of values, obtaining six levels. Thanks to the conception of continuous variation described by Castillo-Garsow (2010), where chunky (or discrete continuous) and smooth continuous variation is given rise to.

Following these ideas of mental actions, of the coordination of values and the conception of the change of an independent quantity, Thompson and Carlson (2017) describe the six levels of covariational reasoning, ensuring that the level of covariation that the student develops implies the conception of the rest of the lower levels. It should be noted that these levels of covariational reasoning keep a partially different approach to those described by Carlson et al. (2002), where they start from mental images, while the levels described by Thompson and Carlson (2017) start from variation descriptors. **Table 1** shows major levels of covariational reasoning.

METHODOLOGY

This research explores the covariational reasoning of third grade middle school students when faced with covariation situations. Additionally, we wished to identify the representations used in their reasoning to respond to each situation. A study with a qualitative approach to covariational reasoning is presented, with a descriptive exploratory scope. Hernández-Sampieri et al. (2014) argue about exploratory studies that allow "obtaining information about the possibility of carrying out more complete information regarding a particular context" (p. 91). Although there are registers of covariational studies, the interest is in what the student manages to communicate with his or her prior knowledge, representations, and solution tools. About the descriptive scope, Hernández-Sampieri et al. (2014) comment that "they intend to measure or collect information independently or jointly about the concepts or variables to which they refer" (p. 92) This information is collected from the answers of the applied test. The characteristics of the work lie in the natural environment, inductive data analysis is considered, that is, according to Francis Bacon, to make observation of the phenomena present in the students' answers and on this the global covariational reasoning evidenced is described (Bacon, cited in Rodríguez & Pérez, 2017), since it is planned to classify, from each student, the representation associated with the levels of covariational reasoning found in the items.

Participants

This research was conducted with middle school students in the State of Puebla, Mexico. The population was selected considering the decision of the schools to agree to work in the study, with the permissions and decisions of the principals, specific groups of students were assigned, as well as the times to intervene in the application of the activities. These reasons shape the research to convenience sampling, that is, the samples are formed with the available cases to which there is access (Hernández-Sampieri et al., 2014). The availability and access of the schools was the factor that defined the number of participants in the study, the test was applied to 40 students, 17 women and 23 men, five 13-year-old students, 32 14-year-old students and three 15-year-old students. In addition, a population of students who completed the third grade of middle school in 2023 was sought, since it was considered, with the study programs, that the students knew graphic and tabular representations. They had been taught topics of proportionality and probability, in fact, they had already interacted in the previous course with the concept of variable, they knew how to generate linear equations and correlate variables in the topic of proportionality. It is presumed that each student had faced, through dynamic situations, how to work with variables. That is, it is asserted that the student had interacted at previous levels with representations of the relationship between variables in concrete and abstract environments, which allows imagining these standardized representations as the first viable way to respond to the situation. However, there is the alternative that students may resort to other immediate representations when faced with the need to know the relationship between variables, resulting in a different representation of covariational reasoning.

Data Collection Instruments

The research data were obtained from the students' answers to a series of questions arising from a covariational situation in which the dynamic variables of the problem had to be identified to find the solution. A certain amount of time was dedicated to the application of the test to keep the students informed that the analysis of the situation required strategies and total concentration on the test. The answers registered were analyzed by the researcher with the description of the levels of covariational reasoning.

The objectives of each situation and the purpose of the items are presented below; the appendix presents the test applied to the students.

Situation 1. Charging a cell phone

In this problem, a comparison is made between two percentage display formats, a graphical representation, and a numerical representation, both expressing the charge of a cell phone. The student had to work with both representations and understand how the amount of charge is expressed in each format; the charge is the same in both formats. The student was to interpret this charge relationship and the format in a continuous or discrete manner. The first four items explore the relationship between numerical charging and graphical charging, in the last three items students are asked to think about both representations and anticipate the charge values in both representations. This situation is explained through the comparison of diagrams. Although no numerical relationship is presented, it is possible to identify a numerical charge format and a visual format whose numerical representation can be obtained with intervals, the student should take this in consideration to ensure a better level of reasoning about the behavior of variables.

Situation 2. The bottle problem

The first item explores the student's reasoning about the shape of the bottles and the speed with which they are filled (one liter in both). In the second item the representation of the change in the level of liquid as it is filled is sought. With the last item a comparison between the amount of liquid in both bottles when the rectangular shape is filled halfway is explored. This to obtain in the student's reasoning the effect that a change in the width of the bottle design generates with respect to a straight vertical design. This situation does not provide numerical values or intervals, it searches for a reasoning in the general vision of the shape of the bottles and the height at which one bottle is filled with respect to the other.

Situation 3. Racing car

With the first item, the initial description seems to be linear in the relationship between distance and time of the vehicle. This changes in the following items, where the student must update the behavior of the vehicle to adjust the speed and predict how the behavior will be in the future, considering that every minute the change is quadratic. The student must build this behavior until obtaining the general representation of the vehicle, in the last item this speed and its change during the registered trajectory is questioned. Unlike the previous situations, this case is described numerically to the student, the relationship between the distance and time traveled by the vehicle is explained with pairs of values, as a problem of proportions.

Data Analysis

The answers of the test applied to the students were analyzed with the levels of covariational reasoning proposed by Thompson and Carlson (2017). For each situation, the students' representations were analyzed in depth to identify patterns in the answers, characteristics of the schemas or arguments, as well as the limitations of the reasoning. The representations made by the students are subject to a visual and spatial image, that is, the appearance of the object and the relationship of the parts of the object to the environment or its movement, respectively. Hegarty and Kozhevnikov (1999) classify these visual-spatial images of students into pictorial and schematic. Pictorial imagery refers to the construction of vivid and detailed visual images. On the other hand, schematic imagery consists of representing the spatial relationships of objects, these categories are described below, considering the answers registered in the study:

1. **Pictorial.** Description of the situation through drawings, the change or transition of values of the variables involved are described as part of the drawing through lines, arrows, marks or symbols (numerical and non-numerical).
2. **Schematic.** The situation is represented by a main drawing accompanied by a numerical description. This description is intended to show the change of values; a series of drawings expressing a transition of states can also be represented. The diagrams can be abstract and represent the change by graphics.

Despite being a type of representation to be considered, the algebraic representation has been discarded in the categorization as it was not present in the participants' answers. Something similar occurs for the verbal representation, since in the situation, brief explanations of the change of the variables are addressed with the help of the items.

Procedure

The research process was developed in three stages, which are described below:

Stage 1. Studies on middle school and high school students were analyzed as a reference of covariational reasoning, as well as the models applied, and strategies used to collect enough data for the analysis.

Stage 2. We worked with 40 participants in third grade of high school. A test was applied with one hour of duration and collected the answers of each situation.

Stage 3. The students' responses were classified according to the level of covariational reasoning, then a category was assigned according to the spontaneous representation used.

RESULTS

The results of the study are presented for each situation presented. The levels of covariational reasoning and the global characteristics of the students' spontaneous representations are described. For this reason, categories or levels absent in the applied tests were not tabulated.

Situation 1. Charging a Cell Phone

The levels of covariational reasoning detected are shown. At the level of pre-coordination of values, the students affirm that the battery charging is faster with the four-bar format than with percentages, the pictorial representation predominates, the drawings show a battery with more or less than four bars (part a in **Figure 1**), others do not explain the change in the charge level (part c in **Figure 1**), some associate a specific percentage with each fragment of the battery (part a in **Figure 1**) but claim that they have a faster charge with the bar format. At the level of gross coordination of values, students represent the interval from one to 100 with load bars (part d in **Figure 1**), the pictorial representation predominates, as they design images that relate the four specific load bars with the load percentages, such as 25, 50 and 100 (part e in **Figure 1**).

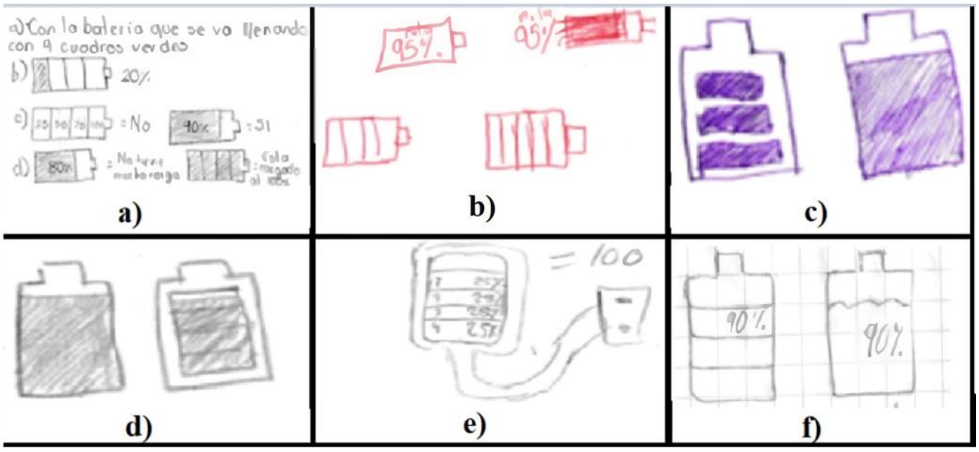

Figure 1. Examples of representations of situation 1 (Source: Field study)

At the level of chunky continuous covariation, the student correlates the loading values for both formats, regardless of the format, the loading level is represented pictorially (part f in **Figure 1**), they respect the loading formats, either the percentage or with the four bars have no problem describing the charge level with images.

Situation 2. The Bottle Problem

The levels of covariational reasoning detected are shown. At the level of pre-coordination of values, students find a relationship between the speed with which the bottles are filled and the width, they affirm that a bottle fills faster. Some claim that the capacity is different depending on the shape, others claim that one bottle is filled and then the other until the heights are equal. The predominant representation is pictorial, the students draw both bottles, some include the capacity (part a in **Figure 2**), but there is no explanation (part b in **Figure 2**). Some drawings depict equal heights but claim that one fills faster than the other (part c in **Figure 2**). At the level of gross coordination of values, the image is of the bottles gradually filling and they ensure that they are filled at the same time, but halfway through, one is fuller, the predominant representation is pictorial, they draw the bottles and the level of liquid at the same height in both (part d in **Figure 2**). A student describes with a sequence of images the level of liquid in both bottles, equalizing the heights (part e in **Figure 2**).

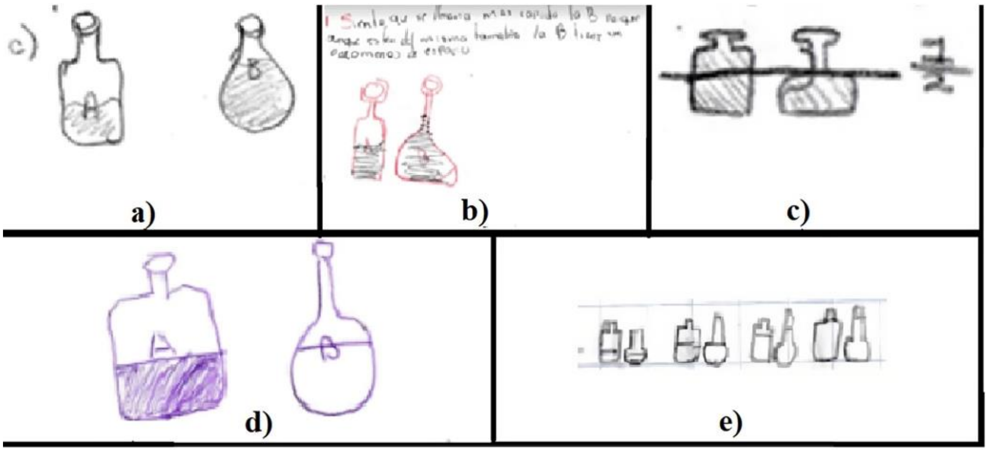

Figure 2. Examples of representations of situation 2 (Source: Field study)

Situation 3. Racing Car

The levels of covariational reasoning detected are shown. In the first level no coordination, the student describes the speed of the vehicle only with the distance traveled, the change in speed is not related to the distance traveled, some vary minutes and kilometers independently. The representation is pictorial, at the level of pre-coordination of values, students describe the relationship between minutes and kilometers, adjust the situation to a linear relationship or invert the dependent variable with the independent one, the representation that prevails is pictorial (part b in **Figure 3**). Another representation is schematic, since students represent the relationship between minutes and kilometers on a Cartesian graph (part c in **Figure 3**), even if the line does not fit the coordinate points. At the level of gross coordination of values, students identify the proportional change between kilometers and minutes, use images of the vehicle to describe the movement (part e in **Figure 3**), numerical description is required, a student schematically represents the drawing of a Cartesian plane with bars (part d in **Figure 3**), at the level of coordination of values, the students distinguish two different events in the situation, the representations are schematic, since the students designed a Cartesian plane and discretized the function (part f in **Figure 3**).

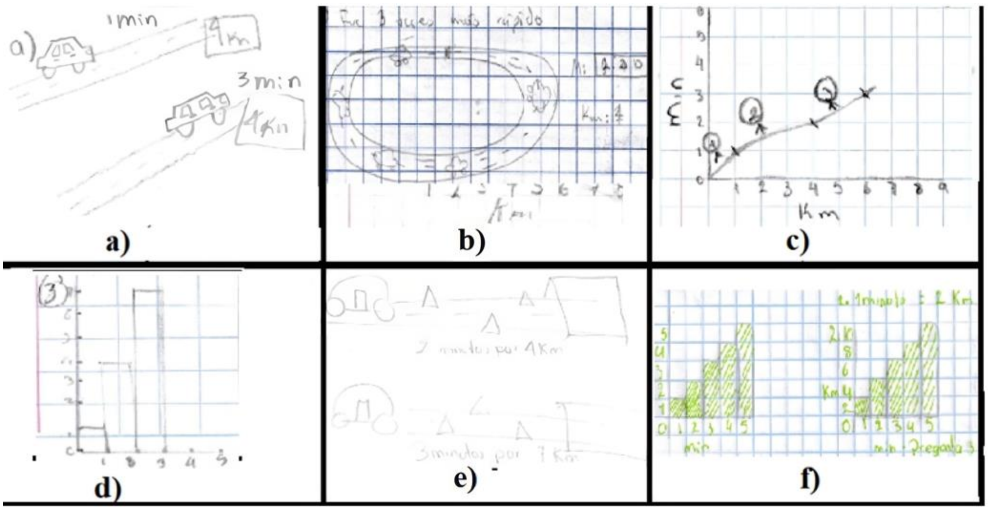

Figure 3. Examples of representations of situation 3 (Source: Field study)

CONCLUSIONS

The described results of the covariational reasoning of middle school students are ordered according to the levels described by Thompson and Carlson (2017), in addition, the spontaneous representations that arise are treated together to determine which representation is most predominant at each level of covariational reasoning. Promoting the reasoning of correspondence between variables in familiar environments can improve the manipulation of variables under appropriate conditions. In this work, it was identified that many third-grade middle school students presented a level below value coordination, that is, they superficially recognized the variables with which they had to work, identified sufficient data, and related them to past cases to adjust the variables to the situation. Even to force the situation to adapt to the selected values, as occurred in situation 3, where they adjusted the problem to two events to preserve the line in the Cartesian plane, these relationships between variables that are expressed in a graph can be found with more frequently at high levels of covariational reasoning, such as the results obtained by Gantt et al. (2022) and Rolfes et al. (2021), in addition, this reasoning is expressed under the same terms of relationship and comparison found by the researchers, with the corresponding spontaneous representation. There are students who did not manage to relate the variables, they visualized independent changes. The students who managed to create this correspondence came to take the wrong variables or assign values outside the conditions of the problem, although this did not prevent them from reaching an elevated level of covariational reasoning, escaping from the conditions of the problem prevents and limits the resolution strategies.

Concerning the spontaneous representations of the students, no algebraic description of variables or any representation of situations with equations was identified. Unlike studies, where it is possible to find the student's level of reasoning with the help of their graphs, a link is even described between the analysis and the development of equations, or using equations to obtain quantities that describe the problem (Gantt et al., 2022; Syarifuddin et al., 2020; Thompson & Carlson, 2017), the results obtained show such reasoning with the help of the student's diagrams, beyond an algebraic representation, an implicit arithmetic explanation is achieved in the drawings made, that is, despite the few values provided in the situations, they recognize the existence of two implicit variables whose values change.

The most obvious representation was the use of static drawings or patterns. The student's intention to explain the behavior of the variables in the situations was linked to a series of images showing how the objects present change, attaching signs or numerical indicators when necessary. A general relationship was found between students who produced static and unexplained drawings with the first two levels of covariational reasoning. It seems that the absence of drawing is linked to the limited understanding of the behavior of variables.

Students who placed themselves in gross coordination and coordination of values levels recognized the relationship between values and made a general description using graphs. Some plotted the Cartesian plane but did not give a tabular representation. It was observed that there is confusion with the use of function graphs and statistical graphs, although the intention of relating pairs of values is superficially understood. They associated such graphs with known cases, since it seems that the numerical description of the situation, as in situation 3, led them to associate known quantities and create a schema of such data.

It could be thought that the rigor of the student's spontaneous representation has a close relationship with the level of covariational reasoning, however, in this work only a general relationship was identified. From this general relationship it can be argued that students with a low level of reasoning covariation used drawings, and at higher levels, they began to use graphs. Due to the time spent working with the student population, the number of questions posed is considered a limitation for the study. For the application, three activities were developed with a limited number of questions due to the time available for class, with the questions they collected sufficient data, but the characteristics of the representations of all the students were not delved into, and it was not possible to consult the students for a greater description of the drawings and complementary elements. Although there was no further communication with the participants, sufficient information was recovered from the questions posed, such as the amount of numerical data that the student knows, the adequate identification of the variables to be treated, the student's experience with other situations and problems he/she has faced, as well as the level of mathematical formality to be able to communicate the abstraction they perform. The students showed the intention to describe the problem, but the lack of knowledge about representations limited them in the explanation of the problem in a verbal and written sense.

Recommendations

In relation to the limitations explained above, some recommendations are listed that may work for future research to improve data collection, analysis, and application of activities:

- Consider a specific time for students to develop the activities carefully, without worrying about the assigned class time ending.
- Asking more questions that produce a spontaneous representation with more details, slightly modifying the questions of the same situation can encourage a different vision of the entire study group.
- Prepare interviews for students whose representations are more detailed or highlight particular elements; this may provide better evidence of covariational reasoning.
- Consider in the approach of the activities more graphic elements that accompany the explanation of the problem, in addition to the written description that is accompanied by numbers, the drawings could spark a closer and objective representation of the student.

The importance of designing tests that explore students' covariational reasoning beyond graphical description is considered. Through this study it was possible to identify the association of variable values with graphic and tabular environments at the levels of covariational reasoning.

However, the students were able to demonstrate this reasoning with other representations, hence the importance of addressing in future research the exploration of situations with different representations for the data.

For teachers, the use of strategies that allow the student to learn about contextualized problems, where it is possible to identify the variables and data, could be an opportunity to improve the level of covariation reasoning in introductory grades, where it is still possible to address covariation situations without a mathematical rigor.

Author contributions: Both authors have sufficiently contributed to the study and agreed with the results and conclusions.

Funding: No funding source is reported for this study.

Acknowledgments: The authors would like to thank CONAHCYT for the financial support to carry out the research studies.

Ethical statement: The authors stated that all research procedures were carried out in accordance with ethical standards and principles of academic integrity. The study focuses on the exploration of the covariational reasoning of students, no clinical interventions, medical experiments or activities that require the ethical requirement of medical or psychological work were involved. No intervention was made in aspects that compromise the health or well-being of the participants. The authors further stated that consent was obtained from the participants before the research was conducted. Participants were informed about the research, anonymity and their rights about withdrawing from the study without adverse repercussions. Data was treated confidentially and respecting the privacy of those present, no sensitive personal data that required the express consent of the participants was collected.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

REFERENCES

Bagossi, S. (2023). Engaging in covariational reasoning when modelling a real phenomenon: The case of the psychometric chart. *Bollettino dell'Unione Matematica Italiana* [*Bulletin of the Italian Mathematical Union*]. https://doi.org/10.1007/s40574-023-00375-7

Bagossi, S., Ferretti, F., & Arzarello, F. (2022). Assessing covariation as a form of conceptual understanding through comparative judgement. *Educational Studies in Mathematics, 111*, 469-492. https://doi.org/10.1007/s10649-022-10178-w

Campos, D. (2010). The imagination and hypothesis-making in mathematics: A Peirce account. In M. Moore (Ed.), *New essays on Peirce's mathematical philosophy* (pp. 123-145). Open Court.

Carlson, M. P. (2002). Physical enactment: A powerful representational tool for understanding the nature of covarying relationships? In F. Hitt (Ed.), *Representations and mathematics visualization* (pp. 63-77). Psychology of Mathematics Education.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 5*(33), 352-378. https://doi.org/10.2307/4149958

Carlson, M. P., Larsen, S., & Jacobs, S. (2001). An investigation of covariational reasoning and its role in learning the concepts of limit and accumulation. In R. Speiser, C. Maher, & C. Walter (Eds.), *Proceedings of the 23rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 145-153). PME-NA.

Carroll, J. B. (1993). *Human cognitive abilities: A survey of factor-analytic studies*. Cambridge University Press. https://doi.org/10.1017/CBO9780511571312

Castillo-Garsow, C. (2010). *Teaching the Verhulst model: A teaching experiment in covariational reasoning and exponential growth* [Unpublished doctoral dissertation]. Arizona State University.

Castillo-Garsow, C. (2012). Continuous quantitative reasoning. In R. Mayes, R. Bonillia, L. L. Hatfield, & S. Belbase (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (pp. 55-73). University of Wyoming Press.

Chen, Y. (2023). *An analysis of covariational reasoning pedagogy for the introduction of derivative in selected calculus textbooks.* [Doctoral dissertation, Columbia University].

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66-86. https://doi.org/10.2307/749228

Copi, I. M. (1978). *Introduction to logic*. Macmillan.

Dwyer, C. A., Gallagher, A., Levin, J., & Morley, M. E. (2003). What is quantitative reasoning? Defining the construct for assessment purposes. *ETS Research Report Series, 2003*(2), i-48. https://doi.org/10.1002/j.2333-8504.2003.tb01922.x

Ellis, A., Ely, R., Singleton, B., & Tasova, H. (2020). Scaling-continuous variation: Supporting students' algebraic reasoning. *Educational Studies in Mathematics, 104*, 87-103. https://doi.org/10.1007/s10649-020-09951-6

Font, V., Godino, J., & D'Amore, B. (2007). An onto-semiotic approach to representations in mathematics education. *For the Learning of Mathematics, 27*(2), 2-14. https://doi.org/10.1007/s11858-006-0004-1

Gantt, A., Paoletti, T., & Vishnubhotla, M. (2022). Constructing a system of covariational relationships: Two contrasting cases. *Educational Studies in Mathematics, 110*(3), 413-433. https://doi.org/10.1007/s10649-021-10134-0

Johnson, H. L. (2015a). Secondary students' quantification of ratio and rate: A framework for reasoning about change in covarying quantities. *Mathematical Thinking and Learning, 17*(1), 64-90. https://doi.org/10.1080/10986065.2015.981946

Johnson, H. L. (2015b). Task design: Fostering secondary students' shifts from variational to covariational reasoning. In K. Beswick, T. Muir, & J. Wells (Eds.), *Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education* (pp. 129-137). University of Tasmania.

Hegarty, M., & Kozhevnikov, M. (1999). Types of visual-spatial representations and mathematical problem solving. *Journal of Educational Psychology, 91*(4), 684-689. https://doi.org/10.1037/0022-0663.91.4.684

Hernández-Sampieri, R., Fernández-Collado, C., & Baptista-Lucio, P. (2014). Definición del alcance de la investigación que se realizará: Exploratorio, descriptivo, correlacional o explicativo [Definition of the scope of the research to be carried out: Exploratory, descriptive, correlational or explanatory]. In R. Hernández-Sampieri, C. Fernández-Collado, & P. Baptista-Lucio (Eds.), *Metodología de la investigación* [*Investigation methodology*] (pp. 88-101). McGraw-Hill.

Hitt, F., & González-Martín, A. S. (2015). Covariation between variables in a modeling process: The ACODESA (collaborative learning, scientific debate and self-reflection) method. *Educational Studies in Mathematics, 88*(2), 201-219. https://doi.org/10.1007/s10649-014-9578-7

Kaput, J. (1995). A research base supporting long term algebra reform? In G. Millsaps, D. Owens, & M. Reed (Eds.), *Proceedings of the 17th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 71-94). ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Martínez-Miraval, M. A., & García-Rodríguez, M. L. (2023). El razonamiento covariacional y su papel en el estudio de la integral definida desde la resolución de problemas [Covariational reasoning and its role in the study of the definite integral from problem solving]. *Tecné, Episteme y Didaxis: TED* [*Tecné, Episteme and Didaxis: TED*], *54*, 154-171. https://doi.org/10.17227/ted.num54-16602

Martínez-Miraval, M. A., García-Cuéllar, D. J., & García-Rodríguez, M. L. (2023). Covariational reasoning and instrumented techniques in the resolution of an optimization problem mediated by GeoGebra. *REDIMAT–Journal of Research in Mathematics Education, 12*(1), 56-81. https://doi.org/10.17583/redimat.11419

Miranda, C., & Sánchez, E. (2019). Framework to analyze and promote the development of functional reasoning in high-school students. In S. Otten, A. G. Candela, Z. de Araujo, C. Haines, & C. Munter (Eds.), *Proceedings of the 41st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 972-979). University of Missouri.

Mkhatshwa, T. (2023). A quantitative and covariational reasoning investigation of students' interpretations of partial derivatives in different contexts. *International Journal of Mathematical Education in Science and Technology, 54*(4), 511-533. https://doi.org/10.1080/0020739X.2021.1958941

Nava-Guzmán, C., García-González, M. S., & Sánchez-Aguilar, M. (2023). Connections between achievement emotions and covariational reasoning: The case of Valeria. *International Electronic Journal of Mathematics Education, 18*(3), em0740. https://doi.org/10.29333/iejme/13180

Oehrtman, M., Carlson, M., & Thompson, P. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. Carlson, & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27-42). Mathematical Association of America. https://doi.org/10.5948/UPO9780883859759.004

Panorkou, N., & Germia, E. F. (2021). Integrating math and science content through covariational reasoning: The case of gravity. *Mathematical Thinking and Learning, 23*(4), 318-343. https://doi.org/10.1080/10986065.2020.1814977

Panorkou, N., & Maloney, A. P. (2016). Early algebra: Expressing covariation and correspondence. *Teaching Children Mathematics, 23*(2), 90-99. https://doi.org/10.5951/teacchilmath.23.2.0090

Passaro, V. (2009). Entre la recherche et la pratique: Regard sur le processus de conception d'une séquence d'enseignement basée sur l'activité mathématique au premier cycle du secondaire [Between research and practice: Look at the process of designing a teaching sequence based on mathematical activity in the first cycle of secondary school]. *Quaderni di Ricerca in Didattica (Matematica)* [*Research Notebooks in Teaching (Mathematics)*], *61*(2), 353-357.

Polya, G. (1984). *Cómo plantear y resolver problemas* [*How to solve it*]. Trillas.

Rodríguez, A., & Pérez, A. O. (2017). Métodos científicos de indagación y de construcción del conocimiento [Scientific methods of inquiry and knowledge construction]. *Revista EAN* [*EAN Magazine*], *82*, 179-200. https://doi.org/10.21158/01208160.n82.2017.1647

Rolfes, T., Roth, J., & Schnotz, W. (2021). Mono- and multi- representational learning of the covariational aspect of functional thinking. *Journal for STEM Education Research, 5*, 1-27. https://doi.org/10.1007/s41979-021-00060-4

Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensah, & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education North America* (pp. 298-304). Raleigh.

Swidan, O., Bagossi, S., Beltramino, S., & Arzarello, F. (2022). Adaptive instruction strategies to foster covariational reasoning in a digitally rich environment. *The Journal of Mathematical Behavior, 66*, 100961. https://doi.org/10.1016/j.jmathb.2022.100961

Syarifuddin, S., Nusantara, T., Qohar, A., & Muksar, M. (2020). Students' thinking processes connecting quantities in solving covariation mathematical problems in high school students of Indonesia. *Participatory Educational Research, 7*(3), 59-78. https://doi.org/10.17275/per.20.35.7.3

Tasova, H., & Moore, K. (2020). Framework for representing a multiplicative object in the context of graphing. In A. I. Sacristán, J. C. Cortés-Zavala, & P. M. Ruiz-Arias (Eds.), *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 236-245). https:/doi.org/10.51272/pmena.42.2020-24

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165-208. https://doi.org/10.1007/BF01273861

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). National Council of Teachers of Mathematics.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4), 356-366. https://doi.org/10.2307/749441

Vygotsky, L. S., Carrasco Iriarte, H., & Ausín, T. (1995). Aproximación al problema [Approach to the problem]. In J. Itzogsohn, J. Piaget, & M. M. Rotger (Eds.), *Pensamiento y lenguaje: Teoría del desarrollo cultural de las funciones* [*Thought and language: theory of cultural development of functions*] (pp. 10-14). La Pléyade.

Yu, F. (2024). Extending the covariation framework: Connecting the covariational reasoning to students' interpretation of rate of change. *The Journal of Mathematical Behavior, 73*, 101-122. https://doi.org/10.1016/j.jmathb.2023.101122